The Impact of Collateral on Swap Rates*

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Abstract

Swap pricing theory traditionally views swaps as portfolios of forward contracts. This intuition breaks down when swaps are marked-to-market and collateralized to mitigate credit exposure which is current market practice. Marking-to-market improves recovery rates and hence affects the default-adjusted rates used to discount swap cash flows. Marking-to-market also results in intermediate cash-flows and it is costly to post collatera. We show that collateralized swap rates differ from those obtained by viewing swaps as portfolio of forward contracts. Ignoring collateralization introduces significant biases in swap rates, especially for long dated swaps.

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1 Introduction

The over-the-counter interest rate swap market has grown exponentially in the last two decades. The notional amount of swaps that is currently outstanding is estimated at nearly $70 trillion US dollars.[1] The rapid growth of this market and the diversity of institutions who take positions in swaps have led to concerns about the integrity of the swap contracts and the potential for a systemic failure arising out of defaults in the swap markets.[2] To mitigate the bilateral counterparty credit exposure, institutions often turn to credit enhancement procedures.[3] The most popular of which is the posting of collateral in the amount of the current mark-to-market (MTM) value of the swap contract (ISDA (1999, page 3).

The traditional approach to swap valuation in Sundaresan (1991) and Duffie and Singleton (1997) views swaps as portfolios of forward contracts on the underlying interest rate. Current market practice requires that swap contracts are marked-to-market and collateralized typically on a daily basis. This credit enhancement results in two important departures from the traditional approach. First, MTM and collateralization generates intermediate payments between the counterparties. Since these cash payments can induce economic costs/benefits to the payer/receiver (either directly or via an opportunity cost of capital), they must be accounted for in valuation. Second, collateralization and MTM change the credit risk exposure of the swap. Collateralization and MTM reduce losses conditional on default and the reduction in credit risk can be dramatic: Collin-Dufresne and Solnik (2000) and He (2001) argue that the market treats swaps as default-free due to collateralization and marking-to-market.

In this paper, we provide a theory of swap valuation when the contracts are collateralized and marked-to-market. We assume that counterparties post U.S. dollar (USD) cash as collateral and mark the contracts to the market value of the contract. Although

[1] See *www.isda.org* for 2001 year end survey results.

[2] See Economist, June 10, 2000 for danger signs in swaps markets.

[3] For a general discussion of these types of procedures, see for example: "An Introduction to Credit Enhancement Techniques," by JP Morgan. For information regarding common market practices, see the "Guidelines for Collateral Practictioners" and "2000 Collateral Survey" from ISDA or "Collateral in wholesale financial markets: trends, risk management and market dynamics" from the Bank for International Settlements for background on market practices.

cash is certainly not the only form of collateral, it is the most popular form of accepted collateral (ISDA (2000), p. 2).[4] In a discrete-time setting, we show that collateralization and marking-to-market result in intermediate cash flows in the swap contract that appear in the form of a stochastic dividend where dividend rate is the cost of posting collateral. This result is similar to the result of Cox, Ingersoll and Ross (1981) who show that the marking-to-marking that occurs in futures contracts results in stochastic dividends.

In continuous-time, we model counterparty credit risk via an exogenous random time τ which indicates default. We follow Duffie and Singleton (1997) and assume that LIBOR is priced using a default-adjusted short rate. Unlike Duffie and Singleton (1997), the occurence of default in the swap and LIBOR markets are not concurrent. In this framework, we prove that the continuous posting of collateral in the mark-to-market value of the swap implies that swap contracts are risk free. Moreover, if there is no cost to posting and maintaining collateral, swaps are priced as in He (2001) and Collin-Dufresne and Solnik (2000) by discounting net swap payments at the risk free rate.

More generally, collateral is not costless and we provide closed form solutions (up to ODE's) for swap rates in the presence of costly collateral when the state variables are affine processes. In this case, it is interesting to note that the cost of posting collateral provides an additional factor that only effects swap rates: it has no impact on default-free securities or on LIBOR. This could provide insight regarding the time series movements of interest rate swap spreads.

Empirically, we show how various assumptions generate different swap curves. Taking the swap curves of Duffie and Singleton (1997) as a benchmark, we show that the swap curve when collateral is costly is higher than the collateralized swap curve when it is costless to post collateral. Next, for typically parameterizations, we show that the costly collateral swap curve is higher than the costless collateral collateral swap curve, but is less than a LIBOR futures contract.

Paramount to understanding the impact of collateralization is the cost of posting collateral. What exactly is this cost? Is it an interest rate spread or just the abstract

[4] Cash collateral may be the cheapest form of collateral in many cases because of the large haircuts required for risky securities and the valuation issues involved in determining the risk in collateral.

oppurtunity cost of capital? To investigate this, we estimate a multi-factor term structure model. The risk-free term structure (estimated from Treasuries) is modeled as a two factor model with the short rate and a time-varying central tendency. This is the same risk-free term structure as Collin-Dufresne and Solnik (2000). We use a single-factor to model the spread between instantaneous LIBOR and Treasuries and also specify that a single factor (which can be correlated with the other three factors) captures the cost of posting cash collateral. These last two factors are extracted from LIBOR and swap rates.

Empirical work to be completed.

Our theory has a number of important implications. First, swap rates are no longer given as par rates off the permanantly refreshed LIBOR curve. Empirical work on swaps rates assume swap rates are par rates off the defaultable LIBOR curve. Recent work using this formulation includes Duffie and Singleton (1997) and Liu, Longstaff and Mandell (2000) who use this formulation in exploring the swap spreads and Piazzesi (2000) who examines the effect of Fed actions on swap rates. If the swap contracts are collateralized, this formulation will, in general, result in a misspecification bias in the empirical results.

Second, we examine the common market practice of bootstrapping the swap curve to obtain zero coupon bond rates. For the LIBOR market, since zero coupon bonds do not typically exist past 1 year, zero coupon bonds prices are typically extracted (bootstrapped) from the par representation of swap rates by interpolating between swap rates.[5] However, with collateralization, this representation no longer holds. Extracting zeros from swap rates to construct forward curves and price interest rate derivatives, is not a valid procedure. In fact, we show that there is not a simple relationship between zeros and swap rates and that one must use a term structure model to extract zeroes (this is similar to the current model-based practice of using a term-structure model to back out forwards from futures data).

[5] We ignore the short end of the curve where futures rates are adjusted in an ad-hoc manner to obtain forward rates.

Last, we examine the pricing of swaptions. If swaps are collateralized, a swaption is not simply the price of an option on a coupon bond with the strike price equal to par.

Section 2 discusses the institutional features of collateralization in the fixed income derivatives market. Section 3 values swaps under collateralization in both discrete and continuous-time. Section 4 considers some theoretical and empirical examples. Section 5 analyzes the effect of collateralization on bootstrapping and the pricing of swaptions. Section 6 concludes.

2 Institutional Features of Collateralization

As mentioned earlier, the rapid growth of the interest rate swap market and the diversity of institutions who take positions in swaps have led to concerns regarding the creditworthiness of counterparties. In order to mitigate their credit exposure, institutions often turn to various credit enhancement procedures. Posting of collateral, either in the form of cash or marketable securities, is the most prominent method of credit enhancement.

It is important to note that while mitigating credit risk is a primary reason for collateralizing derivative transactions, collateral also has other private and social benefits. First, frequent posting and marking-to-market of collateral constrains firms from taking too much leverage, which recently occurred in the market stress of late summer-fall 1998. Second, collateral reduces regulatory capital requirements. According to the Basle accord, collateralized transactions often generate a zero credit risk weighting, which frees up scarce capital for other purposes. Third, using collateral expands the list of potential counterparties as institutions are less concerned about the credit risk of the counterparty provided they are willing to collateralize the transaction. This increases volume and liquidity in swap markets, providing a market wide social benefit of lower spreads and greater competition.

Because of its success, the use of collateral is widespread and growing. The International Swaps and Derivatives Association [ISDA] in their survey of year 2000 estimated that the number of collateralized counterparties is in the range of 1500 to 2500 with a total reported 12,000 signed collateral agreements. The annual growth of collateral agreements was 39% from 1998 to 1999 and projected to increase by 34% from 1999 to

2000. The size of the institutions play a big part in the extent of collateralizing. Large-scale established institutions have collateralized 55% of their OTC derivatives trading [of which swaps are a part] by 1999. Mid-scale established institutions have collateralized 34% of their trades. ISDA (2001) found that more than 65% of swap transactions are collateralized according to the Credit Annex to the Master Swap Agreement. ISDA further estimated that the total collateral in circulation in 1999 was $138 billion.

New institutions such as SWAPCLEAR have been established with the stated purpose of mitigating the credit exposure in the swap markets through large-scale marking-to-market and netting. The development and maintenance of a collateral management program that will support OTC derivatives transactions requires considerable resources. The operating budget for large-scale firms to mount and maintain such a collateral program is estimated by ISDA at US $5 million per annum.

Most of the collateral posted was in the form of USD cash or US government securities, although foreign currencies, major index equities and corporate bonds can also be posted. Together, according to ISDA (2000), cash collateral and U.S. government securities cover about 70% of the posted collateral. Securities whose value changes over time (all collateral accept for USD cash), are more difficult to deal with as the receiver must deal with the risk that the payer will default and the value of the securities posted might fall below the market to market value of the swap. Because of this, non-cash collateral is typically subject to nontrivial haircuts. Due to this, there is an increasing trend toward the use of USD cash collateral.

The collateral no matter what or how it is posted entails a cost. The easiest way to see this is that the receiver of the collateral, when allowed, will typically re-use or re-hypothecate the collateral for other purposes. In fact, according to ISDA, 83% of collateral is reused indicating the economic benefits to the user. For example, the receiver of USD cash can invest it at LIBOR and typically pays the payer less, usually the repo rate. This implies that the receiver of collateral will earn the LIBOR-Repo spread on any received cash collateral. Other securities generate even greater opportunity costs. Treasury and Agency securities can be repo'd out with the collateral holder/payer receiving/losing the benefit. In general, the repo rate on government securities is far greater than the collateral spread and so the opportunity cost of cash collateral will be a lower bound on the opportunity cost.

In a swap contract, the mark-to-market (MTM) procedure typically works as follows. At inception, the swap rate is set so that the present value of all future cash flows is zero. After time passes, however, the present value of the future swap payments varies and the seasoned swap does not generally have zero present value. Because of this, the party whose leg of the swap has negative present value (under water) typically is required to post collateral in the amount equal to the current MTM value of the future swap payments.

Ideally, at each point in time the parties would take an observed market price for the seasoned swap and then mark the contract to market using collateral. In some cases, there may not be market prices available and in this case, the market value is computed using a term structure model.

3 Swap Valuation

This section reviews the traditional approach to swap valuation and provides swap valuation in discrete and continuous-time.

3.1 The Traditional Approach

In this section, we briefly review the traditional approach to swap valuation. We focus on fixed-for-floating interest rate swaps. In this case, Party A pays Party B the fixed swap rate, s_0, and Party B pays Party A a floating rate. In our case, we always assume the floating rate is indexed to 6-month LIBOR. The floating payment can be either the value of the floating rate at the time of the exchange or it can be the value 6-month previous (settled-in-arrears).

To begin, we follow Duffie and Singleton (1997) and assume that discretely compounded LIBOR rates are computed from bond prices that embody default risk. That is, we assert the existence of a risk adjusted instantaneous spot rate, R_t, such that $\tau - period$ LIBOR is given by:

$$L(T,\tau) = \frac{1}{\tau}\left[\frac{1}{P(T,\tau)} - 1\right] \text{ where } P^R(T,\tau) = E_T^Q\left[e^{-\int_T^{T+\tau} R_t dt}\right].$$

We use a subscript to denote time on the state variables and (t, s) to denote current time (t) and contract maturity (s) for prices and yields. The risk-adjusted spot rate is given by $R_t = r_t + \delta_t$ where where r_t is the default-free short rate and δ_t is the spread between instantaneous LIBOR and the default free rate. In Duffie and Singleton (1997), $\delta_t = \lambda_t h_t$ where h_t is the exogenous hazard process and λ_t is the fractional default loss. Intuitively, the probability of default over a short interval Δ is $h_t\Delta$ and $(1 - \lambda_t)$ is the fraction of the market value of the bond recovered conditional on default. Alternatively, we could follow Lando (1996) and assume that there is no recovery which implies that $h_t = 1$.

To value a swap contract, Duffie and Singleton (1997) make the following assumptions:

1. They assume that default risk is exogenous. This implies that λ_t and h_t are not functions of the swap value and are instead functions of exogenous state variables.

2. They assume that both of the swap counterparties have a credit rating equal the average member of the LIBOR panel for the life of the swap.[6]

3. They assume that both parties have the same credit quality. Duffie and Huang (1996) relax this assumption and show that for reasonable assymetric variation in credit quality, there are only small changes in the swap rates.

4. They assume that the default times and recovery rates in the LIBOR and swap markets are the same. As DS (1997) point out, there is no reason to believe that a default event in the over-the-counter derivatives market will coincide with a default event in the LIBOR market.

Of these assumptions, we relax, to some extent, 2, 3 and 4 below. Together, these assumptions imply that swap payments are discounted at R_t. In the case of a single period swap, the market fixed rate solves:

$$E_0^Q \left[e^{-\int_0^T R_t dt} \left(s_0 - L\left(T, \tau\right) \right) \right]$$

[6] Posting of Libor. Actually, for USD swaps, since 1995, 16 banks have been polled the bottom and top four quotes are removed and the others are averaged.

where $L(T,\tau)$ can be $\tau - period$ LIBOR at time T or $\tau - period$ LIBOR at time $T-\tau$. This implies that

$$s_0^R = \frac{E_0^Q\left[e^{-\int_0^T R_t dt} L(T,\tau)\right]}{P^R(0,T)}.$$

When you have a multi-period swap settled-in-arrears on 6-month LIBOR, DS (1997) show that the fixed swap rate is given by:

$$s_0^R = 2\frac{1-P(0,T)}{\sum_{j=1}^{2T} P\left(0,\frac{j}{2}\right)}$$

which is the familiar par rate representation of swap rates. This formulation is used for empirical work, see DS (1997), Piazessi (2001), Liu, Longstaff and Mandel (2002), and is also the basis for pricing swap derivatives as the par representation implies that options on swaps can be viewed as an option on a par-rate coupon bond.

3.2 The Impact of Collateral on Swaps: Discrete Time

In this section, we provide a discrete-time approach to valuing interest rate swaps subject to collateralization. This model-independent formulation draws on the insights of Cox, Ingersoll and Ross (1981). For reasons that will become clear we focus initially on a single period swap that is priced at date 0 for payment at date T. Subsequently, we generalize the result to multi-period swaps with discrete resets and payment dates.

To understand the difference in cash-flows between collateralized and uncollateralized swap transactions, consider the following 2-period example with three dates, $t = 0, 1$ and 2. We assume that USD cash collateralization is used in the swap. At the end of period 2, party A agrees to pay party B a fixed rate and receive the floating rate. In addition, at the end of period 1, the parties agree to mark the swap to market via USD cash collateralization. We assume the parties are symmetrical and that both parties can borrow and lend at LIBOR and the receiver of collateral will credit the payer with the riskless rate (repo). This is consistent with current market practice.

- At time 0, the swap rate, s_0, is set to make the present value of future cash flows zero. Therefore $S_0 = 0$.

- At time 1, suppose the market value of the seasoned swap is $S_1 > 0$. Party B pays Party A $\$S_1$.
- At time 2, Party A receives a benefit from holding the collateral in the amount of $S_1 y_1$; party B entails a cost of posting collateral in the amount of $S_1 y_1$. The swap value is now $S_2 = (l_2 - s_0)$.

Given that collateralization is specified in the original swap agreement, the current swap rate is set to so that the present value of the contract is zero:

$$0 = PV_0 \left[(l_2 - s_0) + S_1 y_1 \right].$$

This is different from the uncollateralized formulation where the swap rate solves

$$0 = PV_0 \left[S_2, 2 \right] = PV_0 \left[LIBOR_2 - x_0, 2 \right],$$

where we intentionally do not specify what interest rate (default-free or default-risky) is used to discount the cash flows.

What is the impact of marking-to-market and collateralization? There are three main implications. First, MTM and collateralization result in a stochastic dividend at the intermediate time-period, $S_1 y_1$. This implies that collateralized swaps are no longer portfolios of forward contracts. This result is reminiscent of the results of Cox, Ingersoll and Ross (1981) who demonstrate that, due to marking to market, futures and forward rates are significantly different.[7] Second, MTM and collateralization will alter the recovery characteristics in the case of default. If Party B defaults on Party A, Party A will lose a maximum of $l_2 - s_0 + S_1 y_1$ which is less than $l_2 - s_0$. Third, as noted in the references earlier, collateralization may reduce the probability that Party B defaults as their leverage has been reduced.

In order to formally incorporate default, the next section formally models default and recovery, both in the LIBOR market and the swaps market.

[7] Collateralized and marked to market swap rates are not the same as a futures rate on Libor, although there is some similarity. The reason is that collateralized swaps are marked to the present value of future cash-flows and not the current futures rate. In addition, when positions are marked to market, the cash can be taken out of the margin account.

3.3 The Impact of Collateral on swaps: Continuous-Time

We now turn to the continuous-time valuation of swaps. Following Duffie and Singleton (1997), we take a reduced form approach to valuing (potentially) defaultable securities. This powerful approach allows us to focus on exactly how various assumptions effect swap valuation in a consistent manner.

We retain Duffie and Singleton's (1997) assumptions regarding default in the LIBOR market and assume that the occurence of default in the swap contract can be represented by a first jump time, τ, of jump process with a (potentially stochastic) intensity h. We let $\mathbf{1}_{[\tau>T]} = 1$ if there is no default by time T. What are the default characteristics of the contract? We assume that upon default, there is no recovery in excess of any collateral posted. (Legally, is this the case). For the interesting cases, this assumptions is of no consequence. Unlike DS (1997), we do not assume that the counterparties are refreshed and remain at LIBOR quality throughout the life of the swap, we do not assume that the swap and LIBOR market share the same default characteristics and we do not directly assume that the counterparties are symmetric (more on this below).

What is the value of a swap in this setting? First, consider the case where it is costless to post and maintain collateral and assume that the contracts are continuously marked and the amount of collateral posted at time τ is given by C_τ. In practice, they are typically marked-to-market at least daily with the option to demand additional collateral in the case of large market moves.[8] In this case, the price of the swap, S_t, is given by the usual default-adjusted discounted value of the cash-flows:

$$S_t = E_t^Q \left[e^{-\int_t^T r_s ds} \Phi_T \mathbf{1}_{[\tau>T]} + e^{-\int_t^\tau r_s ds} C_\tau \mathbf{1}_{[\tau \leq T]} \right]$$

where $\Phi_T = l(T,\tau) - s_0$ and $S_0 = 0$ and s_0 is the fixed swap rate. The first term in the value of the swap, $e^{-\int_t^T r_s ds}\Phi_T$, is the present value of the cashflows conditional on no default and the second component, $e^{-\int_t^\tau r_s ds} C_\tau$, is the present value of the amount received conditional on default prior to time T, C_τ.

If we assume that the collateral is posted in the amount of the marked-to-market value of the collateralized swap, $C_t = S_t$, we have that the swap price process solves (for

[8] See, for example the documentation of marking-to-market at SwapClear.

$t < \tau$)

$$S_t = E_t^Q \left[e^{-\int_t^T r_s ds} \Phi_T \mathbf{1}_{[\tau > T]} + e^{-\int_t^\tau r_s ds} S_\tau \mathbf{1}_{[\tau \le T]} \right]$$

which is just a contract with a random termination time. An application of the law of iterated expectations implies that

$$S_t = E_t^Q \left[e^{-\int_t^T r_s ds} \Phi_T \right].$$

Note that an implication of the pricing formula is that the swap contract contains no credit-risk as recovery, in the event of default, is full. This provides a justification for the formula of Collin-Dufresne and Solnik (2000). Second, note that unless $S_t = C_t$, the contract is not risk free in general and there is residual default risk.

Next, consider the case with costly collateral. If we assume that the instananeous cost to posting collateral is $y_s \ge 0$, then we have that for $t < \tau$

$$S_t = E_t^Q \left[e^{-\int_t^T r_s ds} \Phi_T \mathbf{1}_{[\tau > T]} + e^{-\int_t^\tau r_s ds} C_\tau \mathbf{1}_{[\tau \le T]} \right] + \tag{1}$$

$$E_t^Q \left[\mathbf{1}_{[\tau \le T]} \int_t^\tau e^{-\int_t^s r_u du} y_s C_s ds + \mathbf{1}_{[\tau > T]} \int_t^T e^{-\int_t^s r_u du} y_s C_s ds \right]. \tag{2}$$

The first term is the value of the final swap payments conditional on no default, the second is the value of the collateral seized conditional on default, the third is the present value of the oppurtunity cost of the collateral up to a default time and the last term is the value of the collateral. Assuming that $C_s = S_s$ (the contract is marked to the swap value), we have that

$$S_t = E_t^Q \left[e^{-\int_t^T r_s ds} \Phi_T + \int_t^T e^{-\int_t^s r_u du} y_s S_s ds \right]$$

To see this, substitute in the second expression in (1) for $C_\tau = S_\tau$ and rearrange. This formula is the familiar stochastic dividend formula and implies that

$$S_t = E_t^Q \left[e^{-\int_t^T (r_s - y_s) ds} \Phi_T \right].$$

There are a number of important implications. First, the swap contracts are are default free via the posting of collateral in the MTM value of the swap contract. Since recovery is full, the contract is risk-free swap and is just a contract with a random termination time. Second, if you do not mark to the true price process, the contract has residual risk. In this case, the counterparty grabs the collateral. It is straightforward in this case to model recovery beyond collateral.

At initiation, the value of the swap is zero, $S_0 = 0$ which implies that

$$s_0^{r-y} = \frac{E_t^Q\left[e^{-\int_0^T (r_s - y_s)ds} L(T,\tau)\right]}{E_t^Q\left[e^{-\int_0^T (r_s - y_s)ds}\right]}.$$

3.4 Some Comparative Statics

From the previous sections, we can compare the swap rates of DS (1997), s_0^R (R stands for the risk-adjusted discount rate), the swap rates of Collin-Dufresne and Solnick (2000) and He (2001), s_0^r (r stands for the risk-adjusted discount rate) and the costly collateral swap curve: s_0^{r-y} (r stands for the risk-adjusted discount rate). If we define

$$P^x(0,T) = E_0^Q\left[e^{-\int_0^T x_s ds}\right]$$

we have that single-period swap rates are given by:

$$\begin{aligned}
s_0^R &= E_0^Q\left[l(T,\tau)\right] + \frac{cov_0^Q\left[e^{-\int_0^T R_s ds}, L(T,\tau)\right]}{P^R(0,T)} \\
s_0^r &= E_0^Q\left[l(T,\tau)\right] + \frac{cov_0^Q\left[e^{-\int_0^T r_s ds}, L(T,\tau)\right]}{P^r(0,T)} \\
s_0^{r-y} &= E_0^Q\left[l(T,\tau)\right] + \frac{cov_0^Q\left[e^{-\int_0^T (r_s - y_s)ds} L(T,\tau)\right]}{P^{r-y}(0,T)}
\end{aligned}$$

This shows the close relationship between the price of a futures contract on LIBOR, $F_0^L = E_0^Q\left[l(T,\tau)\right]$, and the swap rates. Note that since the covariance between the interest rate factor and LIBOR is typically negative (although it cannot be signed generically), we have that $s_0^{r-y}, s_0^r, s_0^R < F_0^L$. In the case of the collateralized swap, if the cost/benefit to posting collateral is the the risk free rate, $y_t = r_t$, we have that $s_0^{r-y} = F_0^L$. If $y(t)$ is a nonrandom function of time, then $s_0^{r-y} = s_0^r$. This results does not carry over to the multiperiod case due to a convexity effect. In this case, $s_0^{r-y} > s_0^r$.

This characterization shows that the ordering of the swap rates depends on the covariance of the discount factors with LIBOR. For example,

$$s_0^r - s_0^R = \frac{cov_0^Q\left[e^{-\int_0^T r_s ds}, l\left(T,\tau\right)\right]}{P^r\left(0,T\right)} - \frac{cov_0^Q\left[e^{-\int_0^T R_s ds}, l\left(T,\tau\right)\right]}{P^R\left(0,T\right)},$$

and if we assume that (which is commonly supported in the data),

$$cov_0^Q\left[e^{-\int_0^T R_s ds}, l\left(T,\tau\right)\right] < cov_0^Q\left[e^{-\int_0^T r_s ds}, l\left(T,\tau\right)\right]$$

we have $s_0^r - s_0^R > 0$ (since $P^R < P^r$) . This implies that discounting by r instead of R results in higher swap rates, holding all else equal. Similarly, if[9]

$$cov_0^Q\left[e^{-\int_0^T r_s ds}, l\left(T,\tau\right)\right] < cov_0^Q\left[e^{-\int_0^T (r_s - y_s) ds}, l\left(T,\tau\right)\right]$$

we have that

$$F_0^L > s_0^{r-y} > s_0^r > s_0^R.$$

[9] Consider for the condition

$$cov_0^Q\left[e^{-\int_0^T r_s ds}, l\left(T,\tau\right)\right] < cov_0^Q\left[e^{-\int_0^T (r_s - y_s) ds}, l\left(T,\tau\right)\right].$$

A first order approximation to the exponential $e^x = 1 + x$ implies that

$$\begin{aligned} cov_0^Q\left[1 - \int_0^T r_s ds, l\left(T,\tau\right)\right] &< cov_0^Q\left[1 - \int_0^T r_s ds + \int_0^T y_s ds, l\left(T,\tau\right)\right] \\ &= cov_0^Q\left[1 - \int_0^T r_s ds, l\left(T,\tau\right)\right] + cov_0^Q\left[\int_0^T y_s ds, l\left(T,\tau\right)\right] \end{aligned}$$

and thus we have that $s_0^{r-y} > s_0^r$ if $cov_0^Q\left[\int_0^T y_s ds, l\left(T,\tau\right)\right] > 0$. Since $L\left(T,\tau\right) = \frac{1}{\tau}\left[\frac{1}{P(T,\tau)} - 1\right]$ where $P^R\left(T,\tau\right) = \exp\left(\alpha\left(\tau\right) + \beta^r\left(\tau\right) r_T + \beta^\delta\left(\tau\right)\delta_T\right)$ and β^r,$\beta^\delta < 0$ we have that

$$\begin{aligned} cov_0^Q\left[\int_0^T y_s ds, l\left(T,\tau\right)\right] &\propto cov_0^Q\left[\int_0^T y_s ds, \exp\left(\left(-\beta^r\left(\tau\right)\right) r_T + \left(-\beta^\delta\left(\tau\right)\right)\delta_T\right)\right] \\ &\approx cov_0^Q\left[\int_0^T y_s ds, \left(-\beta^r\left(\tau\right)\right) r_T + \left(-\beta^\delta\left(\tau\right)\right)\delta_T\right]. \end{aligned}$$

Thus if $y_t \uparrow \Rightarrow r_T \uparrow$ and/or $y_t \uparrow \Rightarrow \delta_T \uparrow$ the condition will be satisfied.

We can also represent the collateralized swap rate as: ($S_t\left(s_0^{r-y}\right)$ is the value of a seasoned swap struck at s_0^{r-y})

$$s_0^{r-y} = \frac{E_0^Q\left[\int_0^T e^{-\int_0^t r_s ds} y_t S_t\left(s_0^{r-y}\right)dt + e^{-\int_0^T r_s ds} l\left(T,\tau\right)\right]}{P^r\left(0,T\right))} \tag{3}$$

$$= \frac{E_0^Q\left[e^{-\int_0^T r_s ds} l\left(T,\tau\right)\right]}{P^r\left(0,T\right))} + \frac{E_0^Q\left[\int_0^T e^{-\int_0^t r_s ds} y_t S_t\left(s_0^{r-y}\right)dt\right]}{P^r\left(0,T\right))} \tag{4}$$

$$= s_0^r + + \frac{\left[\int_0^T E_0^Q\left\{e^{-\int_0^t r_s ds} y_t S_t\left(s_0^{r-y}\right)\right\}dt\right]}{P^r\left(0,T\right))} \tag{5}$$

Provided $E_0^Q\left[e^{-\int_0^t r_s ds} y_t S_t\right] > 0$, $s_0^{r-y} > s_0^r$.

4 Empirical

We consider the following model for the risk-free and LIBOR/swap term structure:

$$\begin{aligned}
dr_t &= k_r\left(\theta_t - r_t\right)dt + \sigma_r dW_t^r\left(\mathbb{P}\right)\\
d\theta_t &= k_\theta\left(\theta_\theta - \theta_t\right)dt + \sigma_\theta dW_t^\theta\left(\mathbb{P}\right)\\
d\delta_t &= \kappa_\delta\left(\theta_\delta - \delta_t\right)dt + \sigma_\delta dW_t^\delta\left(\mathbb{P}\right)\\
dy_t &= \left[\kappa_y\left(\theta_y - y_t\right) + \kappa_{r,y} r_t + \kappa_{\delta,y}\delta_t\right]dt + \sigma_y dW_t^y\left(\mathbb{P}\right).
\end{aligned}$$

We assume that all of the Brownian motions are correlated, $corr\left(W_t^i, W_t^j\right) = \rho_{ij}$. Gaussian specifications are common when modeling swap rates, see, e.g., Collin-Dufresne and Solnik (2000), He (2001) and Liu, Longstaff and Mandel (2002). Essentially, the model has a two factor specification for the risk-free term structure consisting of the short rate (r_t) and its long run mean (θ_t) and two factor specification for the LIBOR/swap market where δ_t is the instantaneous spread to LIBOR and y_t is the cost of posting collateral. Note that in our specification y_t only affects swap rates.

We assume that under the Q-measure, we assume that $dW_t^i\left(\mathbb{P}\right) = \frac{\lambda_i\kappa_i}{\sigma_i}dt +$ $dW_t^i\left(\mathbb{Q}\right)$

which implies that

$$
\begin{aligned}
dr_t &= k_r\left(\theta_r^{\mathbb{Q}} - r_r\right)dt + \sigma_r dW_t^r\left(\mathbb{Q}\right)\\
d\theta_t &= k_\theta\left(\theta_\theta^{\mathbb{Q}} - r_t\right)dt + \sigma_\theta dW_t^\theta\left(\mathbb{P}\right)\\
d\delta_t &= \left[\kappa_\delta\left(\theta_\delta^{\mathbb{Q}} - \delta_t\right) + \kappa_{r,\delta} r_t\right]dt + \sigma_\delta dW_t^\delta\left(\mathbb{Q}\right)\\
dy_t &= \left[\kappa_y\left(\theta_y^{\mathbb{Q}} - y_t\right) + \kappa_{r,y} r_t + \kappa_{\delta,y}\delta_t\right]dt + \sigma_y dW_t^y\left(\mathbb{Q}\right)
\end{aligned}
$$

and $\theta_r^Q = \theta_r - \lambda_r$. With six-month resettlement, the various swap rates are given by:

$$
\begin{aligned}
s_0^{Col} &= \frac{\sum_{j=1}^{2N} E_0^Q\left[e^{-\int_0^{\frac{j}{2}}(r_s - y_s)ds} L\left(\frac{j}{2},\tau\right)\right]}{\sum_{j=1}^{2N} E_0^Q\left[e^{-\int_0^{\frac{j}{2}}(r_s - y_s)ds}\right]}\\
s_0^{CD-S} &= \frac{\sum_{j=1}^{2N} E_0^Q\left[e^{-\int_0^{\frac{j}{2}} r_s ds} L\left(\frac{j}{2},\tau\right)\right]}{\sum_{j=1}^{2N} E_0^Q\left[e^{-\int_0^{\frac{j}{2}} r_s ds}\right]}\\
s_0^{DS} &= \frac{\sum_{j=1}^{2N} E_0^Q\left[e^{-\int_0^{\frac{j}{2}} R_s ds} L\left(\frac{j}{2},\tau\right)\right]}{\sum_{j=1}^{2N} E_0^Q\left[e^{-\int_0^{\frac{j}{2}} R_s ds}\right]}
\end{aligned}
$$

The appendix provides expressions for these.

4.1 The Impact of Collateral

To guage the potential impact of collateral, we first consider the parameter estimates from Collin-Dufresne and Solnik (2000) who estimate a two-factor risk-free term structure and a single factor spread process. We assume that $\theta_t = \theta$, that is, we turn off the central tendency factor. We consider a base case for the cost of collateral process of $\kappa_y = 1$; $\theta_y^Q = 0.002$ and $\sigma_y = 0.01$ and $\kappa_y^r = \kappa_y^\delta = \rho_{r,y} = \rho_{\delta,y} = 0$. This implies that the mean cost of collateral is 20 basis points. Figure 1 provides the swap curves for the three cases of interest, s_0^{r-y}, s_0^r, and s_0^R. Figure 2 considers the case where $\kappa_y^r = 0.1$ and Figure 3 considers the case where $\kappa_y^\delta = 0.5$.

There are a number of implications. First, there are significant differences between the swap rates calculated by discounting at r_t, s_0^r, and those calculated by discounting at R, s_0^R. Given that the typical bid-ask spread on a swap is a fraction of a basis point, this implies that collateralization has a significant effect even if there is no cost of posting collateral. For parameterizations in Collin-Dufresne and Solnik (2000), the difference is on the order of 2 basis points at 10 years and 4 basis points at 30 years. For other parameterizations, the effects can be quite large.

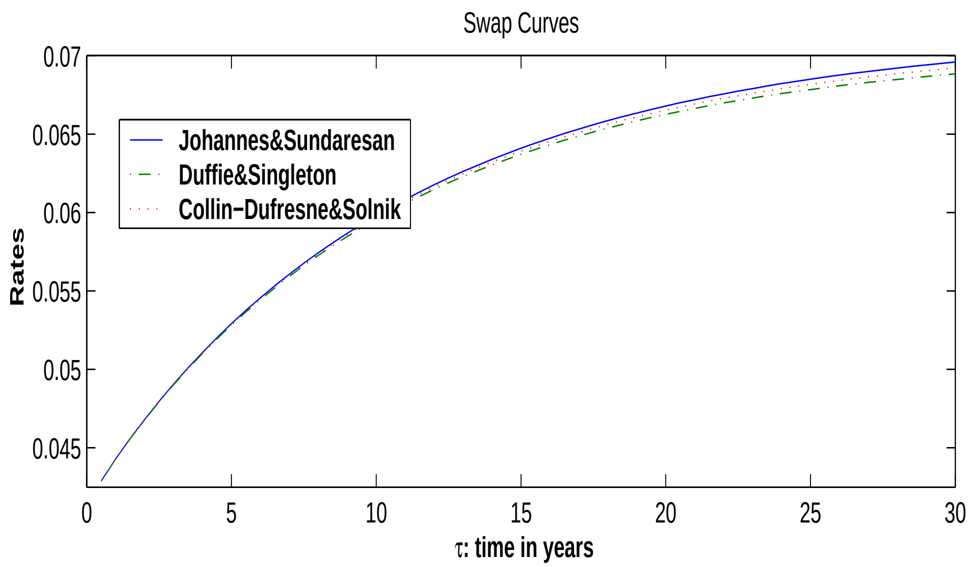
4.2 MLE Estimation

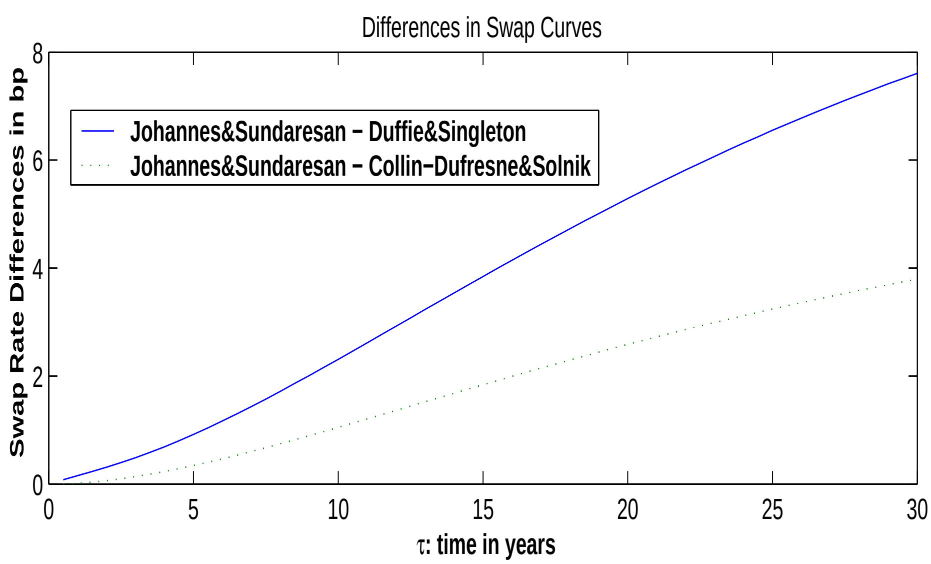
To estimate our model, we consider a two step procedure. In the first stage, we estimate the 2-factor risk-free term structure using time series of 6-month Treasury bill rates and 5, 7 and 10 year par rates. In the second stage, we take these parameters and the state variables as given and estimate a two factor model for the LIBOR/swap market using 6-month LIBOR and 5,7 and 10 year swap rates. The general model is a 4-factor model and we had difficulties obtaining convergence and found the algorithm to be very sensitive to starting values. The two step approach, while sacrificing some efficiency (the extent to which the parameter estimates of the risk-free curve are effected by the swap rates), is a more robust algorithm and has no convergence properties and is less sensitive to starting values.

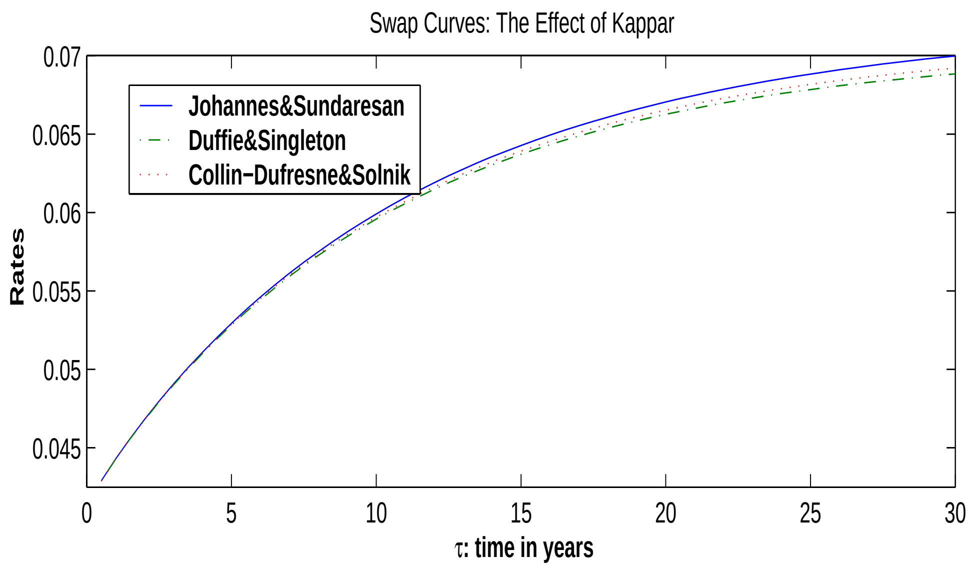
[To be completed]

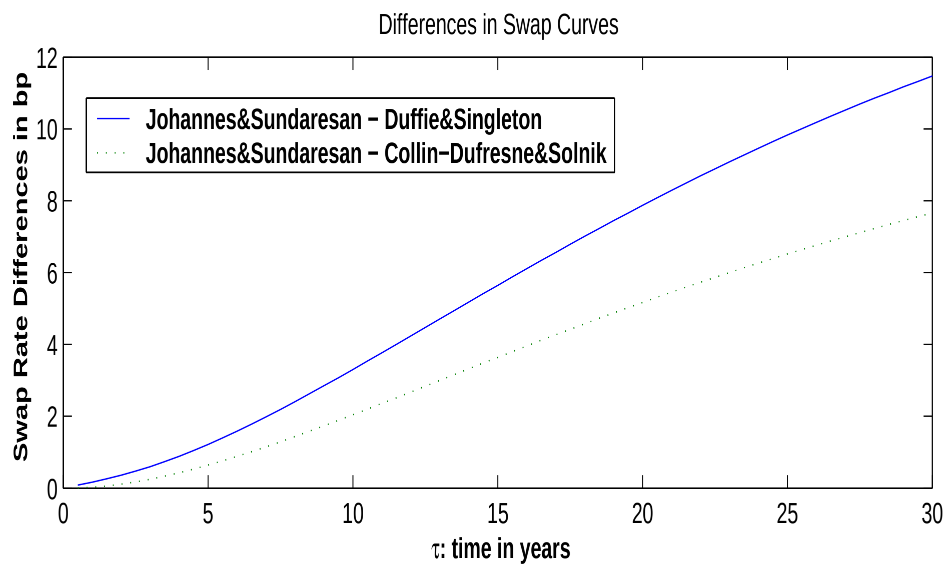
4.3 The Time Series of the Cost of Collateral

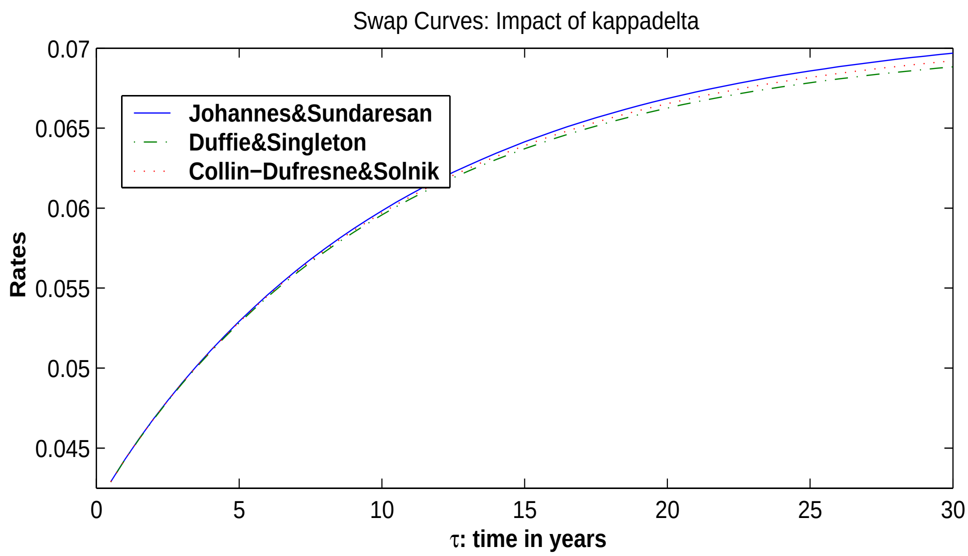
[To be completed]

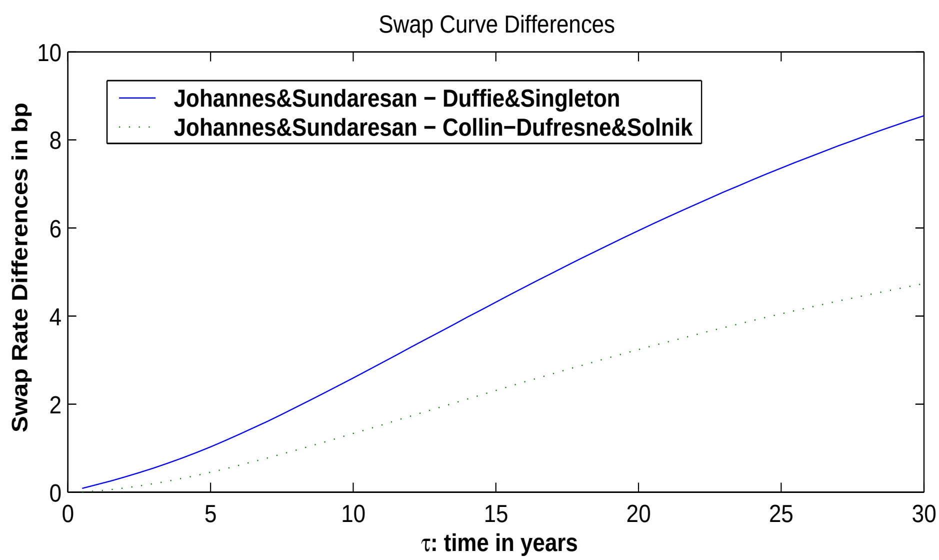

Swap Curves
Rates
0.07
0.065
0.06
0.055
0.05
0.045
Johannes&Sundaresan
Duffie&Singleton
Collin−Dufresne&Solnik
0
5
10
15
20
25
30
τ: time in years


Differences in Swap Curves
Swap Rate Differences in bp
8
6
4
2
0
Johannes&Sundaresan − Duffie&Singleton
Johannes&Sundaresan − Collin−Dufresne&Solnik
0
5
10
15
20
25
30
τ: time in years


Swap Curves: The Effect of Kappar
Rates
0.07
0.065
0.06
0.055
0.05
0.045
0
5
10
15
20
25
30
τ: time in years
Johannes&Sundaresan
Duffie&Singleton
Collin−Dufresne&Solnik


Differences in Swap Curves
Swap Rate Differences in bp
12
10
8
6
4
2
0
0
5
10
15
20
25
30
τ: time in years
Johannes&Sundaresan − Duffie&Singleton
Johannes&Sundaresan − Collin−Dufresne&Solnik


Swap Curves: Impact of kappadelta
Johannes&Sundaresan
Duffie&Singleton
Collin−Dufresne&Solnik
Rates
0.07
0.065
0.06
0.055
0.05
0.045
0
5
10
15
20
25
30
τ: time in years


Swap Curve Differences
Johannes&Sundaresan − Duffie&Singleton
Johannes&Sundaresan − Collin−Dufresne&Solnik
Swap Rate Differences in bp
10
8
6
4
2
0
0
5
10
15
20
25
30
τ: time in years

5 Conclusions

This paper studied the impact of collateralization on swap rates. The impact of collateralization was shown to be very significant when the opportunity costs/benefits of posting collateral are high. The impact is also significant when the risk premium is high. We are currently exploring the consequences of collateralization on extracting zeroes from the swap pricing model and the valuation of swaptions.

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Appendix: ODE's

This appendix derives the ODE's Throughout the appendix, we repeatedly rely on the following useful formula from Duffie, Pan and Singleton (2000): if

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t$$

and $\mu(X_t) = K_0 + K_1 x$, $(\sigma\sigma')_{ij} = (H_0)_{ij} + (H_1)_{ij} \cdot x$, $R(x) = \rho \cdot x$ then for $u \in C^n$, $x \in R^n$, we have that

$$\psi(u, X_t, t, s) = E_t\left[e^{-\int_t^s R(X_u)du} e^{u \cdot X_T}\right] = e^{\alpha(s)+\beta(s)\cdot x}$$

$$\begin{aligned}
\dot{\beta}(t) &= K_1'\beta(t) + \frac{1}{2}\beta(s)' H_1 \beta(s) - \rho \\
\dot{\alpha}(t) &= K_0'\beta(s) + \frac{1}{2}\beta(s)' H_0 \beta(s)
\end{aligned}$$

and $\alpha(0) = \beta(0) = 0$.

Given our model,

$$\begin{aligned}
dr_t &= k_r(\theta_t - r_r)\,dt + \sigma_r dW_t^r(\mathbb{Q}) \\
d\theta_t &= k_\theta\left(\theta_\theta^{\mathbb{Q}} - \theta_t\right)dt + \sigma_\theta dW_t^\theta(\mathbb{P}) \\
d\delta_t &= \kappa_\delta\left(\theta_\delta^{\mathbb{Q}} - \delta_t\right)dt + \sigma_\delta dW_t^\delta(\mathbb{Q}) \\
dy_t &= \left[\kappa_y\left(\theta_y^{\mathbb{Q}} - y_t\right) + \kappa_{r,y} r_t + \kappa_{\delta,y}\delta_t\right]dt + \sigma_y dW_t^y(\mathbb{Q})
\end{aligned}$$

in the form above we have that $X_t' = [r_t, \theta_t, \delta_t, y_t]'$,

$$K_0 = \begin{pmatrix} 0 \\ \kappa_\theta\theta_\theta \\ \kappa_\delta\theta_\delta \\ \kappa_y\theta_y \end{pmatrix},$$

$$K_1 = \begin{pmatrix} -\kappa_r & \kappa_r & 0 & 0 \\ 0 & -\kappa_\theta & 0 & 0 \\ 0 & 0 & -\kappa_\delta & 0 \\ \kappa_y^r & 0 & \kappa_y^\delta & -\kappa_y \end{pmatrix}$$

and

$$H_0 = \begin{pmatrix} \sigma_r^2 & \sigma_r\sigma_\theta\rho_{r,\theta} & \sigma_r\sigma_\delta\rho_{r,\delta} & \sigma_r\sigma_y\rho_{r,y} \\ \sigma_r\sigma_\theta\rho_{r,\theta} & \sigma_\theta^2 & \sigma_\theta\sigma_\delta\rho_{\delta,\theta} & \sigma_\theta\sigma_y\rho_{\delta,y} \\ \sigma_r\sigma_\delta\rho_{r,\delta} & \sigma_\theta\sigma_\delta\rho_{\delta,\theta} & \sigma_\delta^2 & \sigma_\delta\sigma_y\rho_{\delta,y} \\ \sigma_r\sigma_y\rho_{r,y} & \sigma_\theta\sigma_y\rho_{\theta,y} & \sigma_\delta\sigma_y\rho_{\delta,y} & \sigma_y^2 \end{pmatrix}.$$

To compute swap values, we need to compute the following expectations:

$$E_0^Q\left[e^{-\int_0^T x_s ds} L\left(j/2, \tau\right)\right] \text{ and } E_0^Q\left[e^{-\int_0^T x_s ds}\right]$$

for $x_s = R_s$, r_s, $r_s - y_s$.

In this model, 6-month LIBOR is given by $L_t = 2\left[\frac{1}{P^R(t,t+1/2)} - 1\right]$ and

$$P^R\left(t, t+s\right) = E_t^Q\left[e^{-\int_t^{t+s}(r_s+\delta_s)ds}\right] = \exp\left(\alpha_R\left(s\right) + \beta_R\left(s\right)' X_t\right)$$

where $\beta_R = \left[\beta_R^r, \beta_R^\theta, \beta_R^\delta, \beta_R^y\right]$, $\rho_R = [1, 0, 1, 0]$ and $\alpha_R\left(0\right) = \beta_R\left(0\right) = 0$. This implies that $\beta_R^y\left(s\right) = 0$ for all s. Second, we have that

$$P^r\left(t, t+s\right) = E_t^Q\left[e^{-\int_t^{t+s} r_s ds}\right] = \exp\left(\alpha_r\left(s\right) + \beta_r\left(s\right)' X_t\right)$$

$\beta_r = \left[\beta_r^r, \beta_r^\theta, \beta_r^\delta, \beta_r^y\right]$, $\rho_r = [1, 0, 0, 0]$ and we have that $\beta_r^y\left(s\right) = \beta_r^\delta\left(s\right) = 0$ for all s. Third,

$$P^{r-y}\left(t, t+s\right) = E_t^Q\left[e^{-\int_t^{t+s}(r_s-y_s)ds}\right] = \exp\left(\alpha_{r-y}\left(s\right) + \beta_{r-y}\left(s\right)' X_t\right)$$

where $\beta_{r-y} = \left[\beta_{r-y}^r, \beta_{r-y}^\theta, \beta_{r-y}^\delta, \beta_{r-y}^y\left(s\right)\right]$ and $\rho_{r-y} = [1, 0, 0, -1]$.