Financial Numerical Recipes in C++. 

Bernt Arne Ødegaard

June 2014
# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 On C++ and programming.</td>
<td>5</td>
</tr>
<tr>
<td>1.1 Compiling and linking</td>
<td>5</td>
</tr>
<tr>
<td>1.2 The structure of a C++ program</td>
<td>6</td>
</tr>
<tr>
<td>1.2.1 Types</td>
<td>6</td>
</tr>
<tr>
<td>1.2.2 Operations</td>
<td>6</td>
</tr>
<tr>
<td>1.2.3 Functions and libraries</td>
<td>7</td>
</tr>
<tr>
<td>1.2.4 Templates and libraries</td>
<td>7</td>
</tr>
<tr>
<td>1.2.5 Flow control</td>
<td>8</td>
</tr>
<tr>
<td>1.2.6 Input Output</td>
<td>8</td>
</tr>
<tr>
<td>1.2.7 Splitting up a program</td>
<td>8</td>
</tr>
<tr>
<td>1.2.8 Namespaces</td>
<td>9</td>
</tr>
<tr>
<td>1.3 Extending the language, the class concept</td>
<td>9</td>
</tr>
<tr>
<td>1.3.1 date, an example class</td>
<td>10</td>
</tr>
<tr>
<td>1.4 Cost references</td>
<td>16</td>
</tr>
<tr>
<td>1.5 Other C++ concepts</td>
<td>16</td>
</tr>
<tr>
<td>2 Matrix Tools</td>
<td>17</td>
</tr>
<tr>
<td>2.1 The first screen</td>
<td>18</td>
</tr>
<tr>
<td>2.2 Linear algebra</td>
<td>18</td>
</tr>
<tr>
<td>2.2.1 Basic matrix operations</td>
<td>18</td>
</tr>
<tr>
<td>2.2.2 Arithmetic Matrix Operations</td>
<td>19</td>
</tr>
<tr>
<td>2.3 Solving linear equations</td>
<td>22</td>
</tr>
<tr>
<td>2.4 Element by element operations</td>
<td>24</td>
</tr>
<tr>
<td>2.5 Function definitions</td>
<td>24</td>
</tr>
<tr>
<td>2.6 m files</td>
<td>24</td>
</tr>
<tr>
<td>2.7 Flow control</td>
<td>24</td>
</tr>
<tr>
<td>2.8 Plotting</td>
<td>24</td>
</tr>
<tr>
<td>2.9 Libraries</td>
<td>25</td>
</tr>
<tr>
<td>2.10 References</td>
<td>25</td>
</tr>
<tr>
<td>3 The value of time</td>
<td>26</td>
</tr>
<tr>
<td>3.1 Present value</td>
<td>26</td>
</tr>
<tr>
<td>3.2 One interest rate with annual compounding</td>
<td>27</td>
</tr>
<tr>
<td>3.2.1 Internal rate of return</td>
<td>30</td>
</tr>
<tr>
<td>3.3 Continuously compounded interest</td>
<td>34</td>
</tr>
<tr>
<td>3.3.1 Present value</td>
<td>35</td>
</tr>
<tr>
<td>3.4 Further readings</td>
<td>35</td>
</tr>
<tr>
<td>4 Bond Pricing with a flat term structure</td>
<td>36</td>
</tr>
<tr>
<td>4.1 Flat term structure with discrete, annual compounding</td>
<td>37</td>
</tr>
<tr>
<td>4.1.1 Bond Price</td>
<td>37</td>
</tr>
<tr>
<td>4.1.2 Yield to maturity</td>
<td>38</td>
</tr>
<tr>
<td>4.1.3 Duration</td>
<td>41</td>
</tr>
<tr>
<td>4.1.4 Measuring bond sensitivity to interest rate changes</td>
<td>43</td>
</tr>
<tr>
<td>4.2 Continuously compounded interest</td>
<td>47</td>
</tr>
<tr>
<td>4.3 Further readings</td>
<td>50</td>
</tr>
<tr>
<td>5 The term structure of interest rates and an object lesson</td>
<td>51</td>
</tr>
<tr>
<td>5.1 The interchangeability of discount factors, spot interest rates</td>
<td>52</td>
</tr>
<tr>
<td>5.2 The term structure as an object</td>
<td>55</td>
</tr>
<tr>
<td>5.2.1 Base class</td>
<td>55</td>
</tr>
<tr>
<td>5.2.2 Flat term structure</td>
<td>57</td>
</tr>
<tr>
<td>5.3 Using the currently observed term structure</td>
<td>58</td>
</tr>
<tr>
<td>5.3.1 Linear Interpolation</td>
<td>59</td>
</tr>
<tr>
<td>5.3.2 Interpolated term structure class</td>
<td>61</td>
</tr>
<tr>
<td>5.4 Bond calculations with a general term structure and continuous</td>
<td>64</td>
</tr>
<tr>
<td>compounding</td>
<td></td>
</tr>
<tr>
<td>6 The Mean Variance Frontier</td>
<td>67</td>
</tr>
<tr>
<td>6.1 Setup</td>
<td>67</td>
</tr>
<tr>
<td>6.2 The minimum variance frontier</td>
<td>69</td>
</tr>
<tr>
<td>6.3 Calculation of frontier portfolios</td>
<td>69</td>
</tr>
<tr>
<td>6.4 The global minimum variance portfolio</td>
<td>72</td>
</tr>
<tr>
<td>6.5 Efficient portfolios</td>
<td>72</td>
</tr>
<tr>
<td>6.6 The zero beta portfolio</td>
<td>73</td>
</tr>
<tr>
<td>6.7 Allowing for a riskless asset</td>
<td>73</td>
</tr>
<tr>
<td>6.8 Efficient sets with risk free assets</td>
<td>74</td>
</tr>
<tr>
<td>6.9 Short-sale constraints</td>
<td>75</td>
</tr>
<tr>
<td>6.10 The Sharpe Ratio</td>
<td>75</td>
</tr>
<tr>
<td>6.11 Equilibrium: CAPM</td>
<td>76</td>
</tr>
<tr>
<td>6.11.1 Treynor</td>
<td>76</td>
</tr>
<tr>
<td>6.11.2 Jensen</td>
<td>76</td>
</tr>
<tr>
<td>6.12 Working with Mean Variance and CAPM</td>
<td>76</td>
</tr>
<tr>
<td>6.13 Mean variance analysis using matrix libraries</td>
<td>77</td>
</tr>
<tr>
<td>7 Futures algorithms</td>
<td>81</td>
</tr>
<tr>
<td>7.1 Pricing of futures contract</td>
<td>81</td>
</tr>
<tr>
<td>8 Binomial option pricing</td>
<td>82</td>
</tr>
<tr>
<td>8.1 Options</td>
<td>82</td>
</tr>
<tr>
<td>8.2 Pricing</td>
<td>82</td>
</tr>
<tr>
<td>8.3 Multi period binomial pricing</td>
<td>85</td>
</tr>
<tr>
<td>9 Basic Option Pricing, the Black Scholes formula</td>
<td>89</td>
</tr>
<tr>
<td>9.1 The formula</td>
<td>90</td>
</tr>
<tr>
<td>9.2 Understanding the why's of the formula</td>
<td>92</td>
</tr>
<tr>
<td>9.2.1 The original Black Scholes analysis</td>
<td>93</td>
</tr>
<tr>
<td>9.2.2 The limit of a binomial case</td>
<td>93</td>
</tr>
<tr>
<td>9.2.3 The representative agent framework</td>
<td>93</td>
</tr>
<tr>
<td>9.3 Partial derivatives</td>
<td>93</td>
</tr>
<tr>
<td>9.3.1 Delta</td>
<td>93</td>
</tr>
<tr>
<td>9.3.2 Other Derivatives</td>
<td>94</td>
</tr>
<tr>
<td>9.3.3 Implied Volatility</td>
<td>96</td>
</tr>
<tr>
<td>9.4 References</td>
<td>98</td>
</tr>
<tr>
<td>10 Warrants</td>
<td>99</td>
</tr>
<tr>
<td>10.1 Warrant value in terms of assets</td>
<td>99</td>
</tr>
<tr>
<td>10.2 Valuing warrants when observing the stock value</td>
<td>100</td>
</tr>
<tr>
<td>10.3 Readings</td>
<td>101</td>
</tr>
<tr>
<td>11 Extending the Black Scholes formula</td>
<td>102</td>
</tr>
<tr>
<td>11.1 Adjusting for payouts of the underlying</td>
<td>102</td>
</tr>
<tr>
<td>11.1.1 Continuous Payouts from underlying</td>
<td>102</td>
</tr>
<tr>
<td>11.1.2 Dividends</td>
<td>103</td>
</tr>
<tr>
<td>11.2 American options</td>
<td>104</td>
</tr>
<tr>
<td>11.2.1 Exact american call formula when stock is paying one dividend</td>
<td>105</td>
</tr>
<tr>
<td>11.3 Options on futures</td>
<td>108</td>
</tr>
<tr>
<td>11.3.1 Black's model</td>
<td>108</td>
</tr>
<tr>
<td>11.4 Foreign Currency Options</td>
<td>109</td>
</tr>
<tr>
<td>11.5 Perpetual puts and calls</td>
<td>110</td>
</tr>
<tr>
<td>Chapter</td>
<td>Title</td>
</tr>
<tr>
<td>---------</td>
<td>----------------------------------------------------------------------</td>
</tr>
<tr>
<td>11</td>
<td>Readings</td>
</tr>
<tr>
<td>12</td>
<td>Option pricing with binomial approximations</td>
</tr>
<tr>
<td>12.1</td>
<td>Introduction</td>
</tr>
<tr>
<td>12.2</td>
<td>Pricing of options in the Black Scholes setting</td>
</tr>
<tr>
<td>12.2.1</td>
<td>European Options</td>
</tr>
<tr>
<td>12.2.2</td>
<td>American Options</td>
</tr>
<tr>
<td>12.2.3</td>
<td>Matlab implementation</td>
</tr>
<tr>
<td>12.3</td>
<td>How good is the binomial approximation?</td>
</tr>
<tr>
<td>12.3.1</td>
<td>Estimating partials</td>
</tr>
<tr>
<td>12.4</td>
<td>Adjusting for payouts for the underlying</td>
</tr>
<tr>
<td>12.5</td>
<td>Pricing options on stocks paying dividends using a binomial approx-</td>
</tr>
<tr>
<td></td>
<td>imation</td>
</tr>
<tr>
<td>12.5.1</td>
<td>Checking for early exercise in the binomial model</td>
</tr>
<tr>
<td>12.5.2</td>
<td>Proportional dividends</td>
</tr>
<tr>
<td>12.5.3</td>
<td>Discrete dividends</td>
</tr>
<tr>
<td>12.6</td>
<td>Option on futures</td>
</tr>
<tr>
<td>12.7</td>
<td>Foreign Currency options</td>
</tr>
<tr>
<td>12.8</td>
<td>References</td>
</tr>
<tr>
<td>13</td>
<td>Finite Differences</td>
</tr>
<tr>
<td>13.1</td>
<td>Explicit Finite differences</td>
</tr>
<tr>
<td>13.2</td>
<td>European Options</td>
</tr>
<tr>
<td>13.3</td>
<td>American Options</td>
</tr>
<tr>
<td>13.4</td>
<td>Implicit finite differences</td>
</tr>
<tr>
<td>13.5</td>
<td>An example matrix class</td>
</tr>
<tr>
<td>13.6</td>
<td>Finite Differences</td>
</tr>
<tr>
<td>13.7</td>
<td>American Options</td>
</tr>
<tr>
<td>13.8</td>
<td>European Options</td>
</tr>
<tr>
<td>13.9</td>
<td>References</td>
</tr>
<tr>
<td>14</td>
<td>Option pricing by simulation</td>
</tr>
<tr>
<td>14.1</td>
<td>Simulating lognormally distributed random variables</td>
</tr>
<tr>
<td>14.2</td>
<td>Pricing of European Call options</td>
</tr>
<tr>
<td>14.3</td>
<td>Hedge parameters</td>
</tr>
<tr>
<td>14.4</td>
<td>More general payoffs. Function prototypes</td>
</tr>
<tr>
<td>14.5</td>
<td>Improving the efficiency in simulation</td>
</tr>
<tr>
<td></td>
<td>14.5.1 Control variates</td>
</tr>
<tr>
<td></td>
<td>14.5.2 Antithetic variates</td>
</tr>
<tr>
<td>14.6</td>
<td>More exotic options</td>
</tr>
<tr>
<td>14.7</td>
<td>References</td>
</tr>
<tr>
<td>15</td>
<td>Pricing American Options – Approximations</td>
</tr>
<tr>
<td>15.1</td>
<td>The Johnson (1983) approximaion</td>
</tr>
<tr>
<td>15.2</td>
<td>An approximation to the American Put due to Geske and Johnson (1984)</td>
</tr>
<tr>
<td>15.3</td>
<td>A quadratic approximation to American prices due to Barone-Adesi and</td>
</tr>
<tr>
<td></td>
<td>Whaley</td>
</tr>
<tr>
<td>15.4</td>
<td>An alternative approximation to american options due to Bjerksund</td>
</tr>
<tr>
<td></td>
<td>and Stensland (1993)</td>
</tr>
<tr>
<td>15.5</td>
<td>Readings</td>
</tr>
<tr>
<td>16</td>
<td>Average, lookback and other exotic options</td>
</tr>
<tr>
<td>16.1</td>
<td>Bermudan options</td>
</tr>
<tr>
<td>16.2</td>
<td>Asian options</td>
</tr>
<tr>
<td>16.3</td>
<td>Lookback options</td>
</tr>
<tr>
<td>16.4</td>
<td>Monte Carlo Pricing of options whose payoff depend on the whole</td>
</tr>
<tr>
<td></td>
<td>price path</td>
</tr>
<tr>
<td>16.4.1</td>
<td>Generating a series of lognormally distributed variables</td>
</tr>
<tr>
<td>16.5</td>
<td>Control variate</td>
</tr>
<tr>
<td>16.6</td>
<td>References</td>
</tr>
<tr>
<td>17</td>
<td>Generic binomial pricing</td>
</tr>
<tr>
<td>17.1</td>
<td>Introduction</td>
</tr>
<tr>
<td>17.2</td>
<td>Delta calculation</td>
</tr>
<tr>
<td>18</td>
<td>Trinomial trees</td>
</tr>
<tr>
<td>18.1</td>
<td>Intro</td>
</tr>
<tr>
<td>18.2</td>
<td>Implementation</td>
</tr>
<tr>
<td>18.3</td>
<td>Further reading</td>
</tr>
<tr>
<td>19</td>
<td>Alternatives to the Black Scholes type option formula</td>
</tr>
<tr>
<td>19.1</td>
<td>Merton's Jump diffusion model</td>
</tr>
<tr>
<td>19.2</td>
<td>Exton's pricing formula for a stochastic volatility model</td>
</tr>
<tr>
<td>20</td>
<td>Pricing of bond options</td>
</tr>
<tr>
<td>20.1</td>
<td>Black Scholes bond option pricing</td>
</tr>
<tr>
<td>20.2</td>
<td>Binomial bond option pricing</td>
</tr>
<tr>
<td>21</td>
<td>Credit risk</td>
</tr>
<tr>
<td>21.1</td>
<td>The Merton Model</td>
</tr>
<tr>
<td>21.2</td>
<td>Issues in implementation</td>
</tr>
<tr>
<td>22</td>
<td>Term Structure Models</td>
</tr>
<tr>
<td>22.1</td>
<td>The Nelson Siegel term structure approximation</td>
</tr>
<tr>
<td>22.2</td>
<td>Extended Nelson Siegel models</td>
</tr>
<tr>
<td>22.3</td>
<td>Cubic spline</td>
</tr>
<tr>
<td>22.4</td>
<td>Cox Ingersoll Ross</td>
</tr>
<tr>
<td>22.5</td>
<td>Vasicek</td>
</tr>
<tr>
<td>22.6</td>
<td>Readings</td>
</tr>
<tr>
<td>23</td>
<td>Binomial Term Structure models</td>
</tr>
<tr>
<td>23.1</td>
<td>The Rendleman and Bartter model</td>
</tr>
<tr>
<td>23.2</td>
<td>Readings</td>
</tr>
<tr>
<td>24</td>
<td>Interest rate trees</td>
</tr>
<tr>
<td>24.1</td>
<td>The movement of interest rates</td>
</tr>
<tr>
<td>24.2</td>
<td>Discount factors</td>
</tr>
<tr>
<td>24.3</td>
<td>Pricing bonds</td>
</tr>
<tr>
<td>24.4</td>
<td>Callable bond</td>
</tr>
<tr>
<td>24.5</td>
<td>Readings</td>
</tr>
<tr>
<td>25</td>
<td>Building term structure trees using the Ho and Lee (1986) approach</td>
</tr>
<tr>
<td>25.1</td>
<td>Intro</td>
</tr>
<tr>
<td>25.2</td>
<td>Building trees of term structures</td>
</tr>
<tr>
<td>25.3</td>
<td>Ho Lee term structure class</td>
</tr>
<tr>
<td>25.4</td>
<td>Pricing things</td>
</tr>
<tr>
<td>25.5</td>
<td>References</td>
</tr>
<tr>
<td>26</td>
<td>Term Structure Derivatives</td>
</tr>
<tr>
<td>26.1</td>
<td>Vasicek bond option pricing</td>
</tr>
<tr>
<td>27</td>
<td>Date (and time) revisited - the BOOST libraries</td>
</tr>
<tr>
<td>27.1</td>
<td>References</td>
</tr>
</tbody>
</table>

A Normal Distribution approximations.
A.1 The normal distribution function                     | 231  |
A.2 The cumulative normal distribution                    | 232  |
A.3 Multivariate normal                                   | 232  |
A.4 Calculating cumulative bivariate normal probabilities | 233  |
A.5 Simulating random normal numbers                      | 235  |
A.6 Cumulative probabilities for general multivariate distributions | 236  |
A.7 Implementation in C++                                | 236  |
A.8 References                                          | 236  |
This book is a discussion of the calculation of specific formulas in finance. The field of finance has seen a rapid development in recent years, with increasing mathematical sophistication. While the formalization of the field can be traced back to the work of Markowitz (1952) on investors mean-variance decisions and Modigliani and Miller (1958) on the capital structure problem, it was the solution for the price of a call option by Black and Scholes (1973); Merton (1973) which really was the starting point for the mathematicalization of finance. The fields of derivatives and fixed income have since then been the main fields where complicated formulas are used. This book is intended to be of use for people who want to both understand and use these formulas, which explains why most of the algorithms presented later are derivatives prices.

This project started when I was teaching a course in derivatives at the University of British Columbia, in the course of which I sat down and wrote code for calculating the formulas I was teaching. I have always found that implementation helps understanding these things. For teaching such complicated material it is often useful to actually look at the implementation of how the calculation is done in practice. The purpose of the book is therefore primarily pedagogical, although I believe all the routines presented are correct and reasonably efficient, and I know they are also used by people to price real options.

To implement the algorithms in a computer language I choose C++. My students keep asking why anybody would want to use such a backwoods computer language, they think a spreadsheet can solve all the worlds problems. I have some experience with alternative systems for computing, and no matter what, in the end you end up being frustrated with higher end "languages", such as Matlab og R (Not to mention the straitjacket which is is a spreadsheet.) and going back to implementation in a standard language. In my experience with empirical finance I have come to realize that nothing beats knowledge a real computer language. This used to be FORTRAN, then C, and now it is C++. All example algorithms are therefore coded in C++. I do acknowledge that matrix tools like Matlab are very good for rapid prototyping and compact calculations, and will in addition to C++ in places also illustrate the use of Matlab, as well as other (public domain) tools.

The manuscript has been sitting on the internet a few of years, during which it has been visited by a large number of people, to judge by the number of mails I have received about the routines. The present (2014) version mainly expands on the background discussion of the routines, it is more extensive, (but it does not replace a real textbook). I have also added some introductory material on how to program in C++, since a number of questions make it obvious this manuscript is used by a number of people who know finance but not C++. All the routines have been made to confirm to the new ISO/ANSI C++ standard, using such concepts as namespaces and the standard template library. The latest (2011) C++ standard introduced a few useful simplifications, which is incorporated in places.

The current manuscript therefore has various intended audiences. Primarily it is for students of finance who desires to see a complete discussion and implementation of some formula. But the manuscript is also useful for students of finance who wants to learn C++, and for computer scientists who want to understand about the finance algorithms they are asked to implement and embed into their programs.

In doing the implementation I have tried to be as generic as possible in terms of the C++ used, but I have taken advantage of a some of the possibilities the language provides in terms of abstraction and modularization. This will also serve as a lesson in why a real computer language is useful. For example I have encapsulated the term structure of interest rate as an example of the use of classes.

This is not a textbook in the underlying theory, for that there are many good alternatives. For much of the material the best textbooks to refer to are Hull (2011) and McDonald (2013), which I have used as references. The notation of the present manuscript is also similar to these books.
Chapter 1

On C++ and programming.

Contents

1.1 Compiling and linking .................................................. 5
1.2 The structure of a C++ program ......................................... 6
  1.2.1 Types ................................................................. 6
  1.2.2 Operations ........................................................ 6
  1.2.3 Functions and libraries .......................................... 7
  1.2.4 Templates and libraries ......................................... 7
  1.2.5 Flow control ...................................................... 8
  1.2.6 Input Output ....................................................... 8
  1.2.7 Splitting up a program ........................................ 8
  1.2.8 Namespaces ....................................................... 9
1.3 Extending the language, the class concept. ......................... 9
  1.3.1 date, an example class ....................................... 10
1.4 Const references ..................................................... 16
1.5 Other C++ concepts .................................................. 16

In this chapter I introduce C++ and discuss how to run programs written in C++. This is by no means a complete reference to programming in C++, it is designed to give enough information to understand the rest of the book. This chapter also only discusses a subset of C++, it concentrates on the parts of the language used in the remainder of this book. For really learning C++ a textbook is necessary. I have found Lippman and Lajoie (1998) an excellent introduction to the language. The authoritative source on the language is Stroustrup (1997b).

1.1 Compiling and linking

To program in C++ one has to first write a separate file with the program, which is then compiled into low-level instructions (machine language) and linked with libraries to make a complete executable program. The mechanics of doing the compiling and linking varies from system to system, and we leave these details as an exercise to the reader.

1I learned C++ from the previous edition of the book, Lippman (1992). From what I can tell the present editions still seems like a good way of learning the language, but C++ has changed a lot in recent years.
1.2 The structure of a C++ program

The first thing to realize about C++ is that it is a strongly typed language. Everything must be declared before it is used, both variables and functions. C++ has a few basic building blocks, which can be grouped into types, operations and functions.

1.2.1 Types

The types we will work with in this book are bool, int, long, double and string.

Here are some example definitions:

```cpp
bool this_is_true=true;
int i = 0;
long j = 123456789;
double pi = 3.141592653589793238462643;
string s("this is a string");
```

The most important part of C++ comes from the fact that these basic types can be expanded by use of classes, of which more later.

1.2.2 Operations

To these basic types the common mathematical operations can be applied, such as addition, subtraction, multiplication and division:

```cpp
int i = 100 + 50;
int j = 100 - 50;
int n = 100 * 2;
int m = 100 / 2;
```

These operations are defined for all the common datatypes, with exception of the string type. Such operations can be defined by the programmer for other datatypes as well.

Increment and decrement In addition to these basic operations there are some additional operations with their own shorthand. An example we will be using often is incrementing and decrementing a variable. When we want to increase the value of one item by one, in most languages this is written:

```cpp
int i=0;
i = i+1;
i = i-1;
```

In C++ this operation has its own shorthand

```cpp
int i=0;
i++;
i--;
```

While this does not seem intuitive, and it is excusable to think that this operation is not really necessary, it does come in handy for more abstract data constructs. For example, as we will see later, if one defines a date class with the necessary operations, to get the next date will simply be a matter of

```cpp
date d(1,1,1995);
d++;
```

These two statements will result in the date in d being 2jan95.
1.2.3 Functions and libraries

In addition to the basic mathematical operations there is a large number of additional operations that can be performed on any type. However, these are not parts of the core language, they are implemented as standalone functions (most of which are actually written in C or C++). These functions are included in the large library that comes with any C++ installation. Since they are not part of the core language they must be defined to the compiler before they can be used. Such definitions are performed by means of the include statement.

For example, the mathematical operations of taking powers and performing exponentiation are defined in the mathematical library cmath. In the C++ program one will write

```
#include <cmath>
```

cmath is actually a file with a large number of function definitions, among which one finds pow(x,n) which calculates $x^n$, and exp(r) which calculates $e^r$. The following programming stub calculates $a = 2^2$ and $b = e^1$.

```
#include <cmath>
double a = pow(2,2);
double b = exp(1);
```

which will give the variables a and b values of 4 and 2.718281828..., respectively.

1.2.4 Templates and libraries

The use of libraries is not only limited to functions. Also included in the standard library is generic data structures, which can be used on any data type. The example we will be considering the most is the vector<>, which defines an array, or vector of variables.

```
#include <vector>
vector<double> M(2);
M[0]=1.0;
M[1]=2.0;
M.push_back(3);
```

This example defines an array with three elements of type double

\[
M = \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

Note some peculiarities here. When first defining the vector with the statement

```
vector<double> M(2);
```

we defined an array of 2 elements of type double, which we then proceeded to fill with the values 1 and 2. When filling the array we addressed each element directly. Note that in the statement

```
M[0]=1.0;
```

lies one of the prime traps for programmers coming to C or C++ from another language. Indexing of arrays starts at zero, not at one. $M[0]$ really means the first element of the array.

The last statement,
M.push_back(3);

shows the ability of the programmer of changing the size of the array after it has been defined. push_back is a standard operation on arrays which "pushes" the element onto the back of the array, extending the size of the array by one element. Most programming languages do not allow the programmer to specify variable-sized arrays "on the fly." In FORTRAN or Pascal we would usually have to set a maximum length for each array, and hope that we would not need to exceed that length. The vector<> template of C++ gets rid of the programmers need for "bookkeeping" in such array manipulations.

1.2.5 Flow control

To repeat statements several times one will use on of the possibilities for flow control, such as the for or while constructs. For example, to repeat an operation n times one can use the following for loop:

```c
for (int i=0; i<n; i++) {
    some_operation(i);
}
```

The for statement has three parts. The first part gives the initial condition (i=0). The next part the terminal condition (i<n), which says to stop when i<n is not fulfilled, which is at the n'th iteration. The last part is the increment statement (i++), saying what to do in each iteration. In this case the value of i is increased by one in each iteration. This is the typical for statement. One of the causes of C's reputation for terseness is the possibility of elaborate for constructs, which end up being almost impossible to read. In the algorithms presented in this book we will try to avoid any obfuscated for statements, and stick to the basic cases.

1.2.6 Input Output

For any program to do anything useful it needs to be able to output its results. Input and output operations is defined in a couple of libraries, iostream and fstream. The first covers in/output to standard terminals and the second in/output to files.

To write to standard output cout (the terminal), one will do as follows:

```c
#include <iostream>
cout << "This is a test" << endl;
```

To write to a file "test.out", one will do as follows:

```c
#include <fstream>
ofstream outf;
outf.open("test.out");
outf << "This is a test" << endl;
outf.clear();
outf.close();
```

1.2.7 Splitting up a program

Any nontrivial program in C++ is split into several pieces. Usually each piece is written as a function which returns a value of a given type. To illustrate we provide a complete example program, shown in C++ Code 1.1.
The program defines a function performing the mathematical power operation, \( \text{power}(x, n) \) which calculates \( x^n \) through the identity \( x^n = e^{n \log(x)} \). This function is then used to calculate and print the first 5 powers of 2.

When compiled, linked and run, the program will provide the following output:

\[
2^1 = 2 \\
2^2 = 4 \\
2^3 = 8 \\
2^4 = 16 \\
2^5 = 32
\]

### Namespaces

To help in building large programs, the concept of a namespace was introduced. Namespaces are a means of keeping the variables and functions defined local to the context in which they are used. For now it is necessary to know that any function in the standard C++ library lies in its own namespace, called the standard namespace. To actually access these library functions it is necessary to explicitly specify that one wants to access the standard namespace, by the statement

```
using namespace std;
```

Instead of such a general approach, one can also specify the namespace on an element by element basis, but this is more a topic for specialized C++ texts, for the current purposes we will allow all routines access to the whole standard namespace.

### Extending the language, the class concept.

One of the major advances of C++ relative to other programming languages is the programmers ability to extend the language by creating new data types and defining standard operations on these data types. This ability is why C++ is called an object oriented programming language, since much of the work in programming is done by creating objects. An object is best thought of as a data structure with operations on it defined. How one uses an object is best shown by an example.
1.3.1 date, an example class

Consider the abstract concept of a date. A date can be specified in any number of ways. Let us limit ourselves to the Gregorian calendar. 12 august 2003 is a common way of specifying a date. However, it can also be represented by the strings: “2003/8/12”, “12/8/2003” and so on, or by the number of years since 1 January 1900, the number of months since January, and the day of the month (which is how a UNIX programmer will think of it).

However, for most people writing programs the representation of a date is not relevant, they want to be able to enter dates in some abstract way, and then are concerned with such questions as:

- Are two dates equal?
- Is one date earlier than another?
- How many days is it between two dates?

A C++ programmer will proceed to use a class that embodies these uses of the concept of a date. Typically one will look around for an extant class which has already implemented this, but we will show a trivial such date class as an example of how one can create a class.

```cpp
class date {
protected:
   int year_;
   int month_;
   int day_;
public:
   date();
   date(const int& d, const int& m, const int& y);

   bool valid() const;
   int day() const;
   int month() const;
   int year() const;

   void set_day (const int& d);
   void set_month (const int& m);
   void set_year (const int& y);

   date operator ++(); // prefix
   date operator ++(int); // postfix
   date operator --();  // prefix
   date operator --(int); // postfix
};
```

```cpp
bool operator == (const date&, const date&); // comparison operators
bool operator != (const date&, const date&);
bool operator < (const date&, const date&);
bool operator > (const date&, const date&);
bool operator <= (const date&, const date&);
bool operator >= (const date&, const date&);
```

**Header file 1.1: Defining a date class**

A class is defined in a header file, as shown in Header file 1.1. As internal representation of the date is chosen the three integers day_, month_ and year_. This is the data structure which is then manipulated by the various functions defined below.

The functions are used to
- Create a date variable: `date(const int& d, const int& m, const int& y);
- Functions outputting the date by the three integer functions `day()`, `month()` and `year()`.
- Functions setting the date `set_day(int)`, `set_month(int)` and `set_year(int)`, which are used by providing an integer as arguments to the function.
- Increment and decrement functions `++` and `-`
- Comparison functions `<`, `<=`, `>`, `>=`, `==` and `!=`.

After including this header file, programmers using such a class will then treat an object of type `date` just like any other.

For example,

```cpp
date d(1,1,2001);
++d;
```

would result in the `date` object `d` containing the date 2 January 2001.

Any C++ programmer who want to use this `date` object will only need to look at the header file to know what are the possible functions one can use with a `date` object, and be happy about not needing to know anything about how these functions are implemented. This is the encapsulation part of object oriented programming, all relevant information about the `date` object is specified by the header file. This is the only point of interaction, all details about implementation of the class objects and its functions is not used in code using this object. In fact, the user of the class can safely ignore the class' privates, which is only good manners, anyway.

Let us look at the implementation of this.

C++ `Code 1.2` defines the basic operations, initialization, setting the date, and checking whether a date is valid.
#include "date.h"

date::date(){
    year_ = 0;
    month_ = 0;
    day_ = 0;
};

date::date(
    const int& day,
    const int& month,
    const int& year){
    day_ = day;
    month_ = month;
    year_ = year;
};

int date::day() const {
    return day_;}
int date::month() const {
    return month_;}
int date::year() const {
    return year_;}

void date::set_day(
    const int& day){
    date::day_ = day;
};
void date::set_month(const int& month){
    date::month_ = month;
};
void date::set_year(const int& year){
    date::year_ = year;
};

bool date::valid() const {
    // This function will check the given date is valid or not.
    // If the date is not valid then it will return the value false.
    // Need some more checks on the year, though
    if (year_ < 0)
        return false;
    if (month_ > 12 || month_ < 1) return false;
    if (day_ > 31 || day_ < 1) return false;
    if ((day_ == 31 && (month_ == 2 || month_ == 4 || month_ == 6 || month_ == 9 || month_ == 11))
        return false;
    if (day_ == 30 && month_ == 2) return false;
    // should also check for leap years, but for now allow for feb 29 in any year
    return true;
};
For many abstract types it can be possible to define an ordering. For dates there is the natural ordering. C++ Code 1.3 shows how such comparison operations is defined.

```cpp
#include "date.h"

bool operator == (const date& d1, const date& d2){ // check for equality
    if (! (d1.valid()) && (d2.valid())) { return false; } /* if dates not valid, not clear what to do. alternative: throw exception */
    return ((d1.day()==d2.day()) && (d1.month()==d2.month()) && (d1.year()==d2.year()));
}

bool operator < (const date& d1, const date& d2){
    if (! (d1.valid()) && (d2.valid())) { return false; } // see above remark
    if (d1.year()==d2.year()) { // same year
        if (d1.month()==d2.month()) { // same month
            return (d1.day()<d2.day());
        } else {
            return (d1.month()<d2.month());
        }
    } else { // different year
        return (d1.year()<d2.year());
    }
    return false;
}

// remaining operators defined in terms of the above

bool operator <=(const date& d1, const date& d2){
    if (d1<=d2) { return true; }
    return (d1<d2);
}

bool operator >(const date& d1, const date& d2) {
    if (d1==d2) { return true; }
    return (d1>d2);
}

bool operator >=(const date& d1, const date& d2)
    return !(d1<=d2);

bool operator !=(const date& d1, const date& d2){ return !(d1==d2);
```

C++ Code 1.3: Comparison operators for the date class
C++ Code 1.4 shows operations for finding previous and next date, called an iteration operator.

```cpp
#include "date.h"

date next_date(const date& d)
{
    if (!d.valid()) { return date(); } // first try adding a day
    date ndat = date((d.day() + 1), d.month(), d.year());
    if (ndat.valid()) return ndat;
    ndat = date(1, (d.month() + 1), d.year()); // then try adding a month
    if (ndat.valid()) return ndat;
    ndat = date(1, 1, (d.year() + 1)); // must be next year
    return ndat;
}

date previous_date(const date& d)
{
    if (!d.valid()) { return date(); } // return the default date
    date pdat = date((d.day() - 1), d.month(), d.year());
    if (pdat.valid()) return pdat; // try same month
    pdat = date(31, (d.month() - 1), d.year());
    if (pdat.valid()) return pdat;
    pdat = date(30, (d.month() - 1), d.year());
    if (pdat.valid()) return pdat;
    pdat = date(29, (d.month() - 1), d.year());
    if (pdat.valid()) return pdat;
    pdat = date(28, (d.month() - 1), d.year());
    if (pdat.valid()) return pdat;
    pdat = date(31, 12, (d.year() - 1)); // try previous year
    return pdat;
}

date date::operator ++(int){ // postfix operator
    date d = *this;
    *this = next_date(d);
    return d;
}

date date::operator ++(){ // prefix operator
    *this = next_date(*this);
    return *this;
}

date date::operator --(int){ // postfix operator, return current value
    date d = *this;
    *this = previous_date(*this);
    return d;
}

date date::operator --(){ // prefix operator, return new value
    *this = previous_date(*this);
    return *this;
}
```

C++ Code 1.4: Iterative operators for the date class
Exercise 1.1.
The function valid() in the date class accepts February 29'th in every year, but this should ideally only happen for leap years. Modify the function to return a false if the year is not a leap year.

Exercise 1.2.
A typical operating system has functions for dealing with dates, which your typical C++ implementation can call. Find the relevant functions in your implementation, and

1. Implement a function querying the operating system for the current date, and return this date.

2. Implement a function querying the operating system for the weekday of a given date, and return a representation of the weekday as a member of the set:

   {"mon" , "tue" , "wed" , "thu" , "fri" , "sat" , "sun" }

3. Reimplement the valid() function using a system call.

Exercise 1.3.
Once the date class is available, a number of obvious functions begs to be implemented. How would you

1. Add a given number of days to a date?

2. Go to the end or beginning of a month?

3. Find the distance between two dates (in days or in years)?

4. Extract a date from a string? (Here one need to make some assumptions about the format)

Exercise 1.4.
Take a look at how dates are dealt with in various computing environments, such as the operating system (Unix, Windows), applications (Spreadsheets), programming languages, etc. At what level of abstraction is the interface? Do you need to know how dates are implemented? For those with access to both Matlab and Windows, why would you say that Matlab has an "off-by-one" problem relative to Windows?
1.4 **Const references**

Let us now discuss a concept of more technical nature. Consider two alternative calls to a function, defined by function calls:

```cpp
some_function(double r);
some_function(const double& r);
```

They both are called by an argument which is a double, and that argument is guaranteed to not be changed in the calling function, but they work differently. In the first case a copy of the variable referenced to in the argument is created for use in the function, but in the second case one uses the same variable, the argument is a *reference* to the location of the variable. The latter is more efficient, in particular when the argument is a large class. However, one worries that the variable referred to is changed in the function, which in most cases one do not want. Therefore the *const* qualifier, it says that the function can not modify its argument. The compiler will warn the programmer if an attempt is made to modify such a variable.

For efficiency, in most of the following routines arguments are therefore given as as constant references.

1.5 **Other C++ concepts**

A number of other C++ concepts, such as function prototypes and templates, will be introduced later in particular contexts. They only appear in a few places and is better introduced where they are used.
Chapter 2

Matrix Tools

Being computer literate entails being aware of a number of computer tools and being able to choose the most suitable tool for the problem at hand. Way to many people turns this around, and want to fit any problem to the computer tool they know. The tool that very often is the tool for business school students is a spreadsheet like Excel. Of course, a spreadsheet is very useful for very many business applications. However, it is not the best tool for more computationally intensive tasks.

While the bulk of the present book concerns itself with C++, in many applications in finance a very handy tool is a language for manipulating vectors and matrices using linear algebra. There are a lot of different possible programs that behaves very similarly, with a syntax taken from the mathematical formulation of linear algebra. An early tool of this sort was matlab, with a large number of programs copying much of the syntax of this program. As a result of this there is a proliferation of programs with similar syntax to Matlab doing similar analysis. General tools include the commercial package Matlab sold by Mathworks, the public domain programs octave and scilab. Tools that are similar, but more geared towards econometrics, include Gauss, Ox and S with its public domain “clone” R. As for what program to install, there is no right answer. For the basic learning of how these tools work, any of the mentioned packages will do the job. For students on a limited budget the public domain tools octave, scilab and R are obvious candidates. All of them perform the basic operations done by the commercial Matlab package, and good for learning the basics of such a matrix tool.

All of these tools are programs that lets the user manipulate vectors and matrices using very compact notation. While compact notation is always prone to tense, making programs using it unreadable, this is not such a large problem in Matlab, the notation tends to be so close how a mathematician would write them that programs can be relatively easy to follow. There are some pitfalls for the unwary user, in particular it is easy to “miss” the difference between a command operating on a whole matrix and the corresponding element by element operation. For example, consider the following short example, where the operator `^` means that the matrix $A$ is taken to the power 2 (multiplied with itself), and the operator `.^` means that each element of the matrix $A$ is taken to the power 2. The two commands give very different answers.

```matlab
>> A = [1 1 ; 1 1]
A =
    1 1
    1 1
>> A^2
ans =
    2 2
    2 2
>> A.^2
ans =
```

17
The rest of this chapter gives an introduction to a tool like this.

### 2.1 The first screen

How you start the particular tool you are using depend both on which program and which operating system you are working on. The details of how to start it is left as an exercise to the reader. The tools are interactive, they present you with a prompt, and expect you to start writing commands. We will

\[
\texttt{>>}
\]

as the prompt, which means that the program is ready to receive commands.

In the text output of the matrix tool will be shown typewritten as:

\[
\texttt{>> A = [1, 2, 3; 4, 5, 6]}
\]

This particular command defines a matrix \(A\), the matrix tool will respond to this command by printing the matrix that was just defined:

\[
A = \\
1 2 3 \\
4 5 6
\]

### 2.2 Linear algebra

To use such a tool you need some knowledge of linear algebra. We assume the reader have this basic knowledge, if not a quick perusal of a standard mathematical text on linear algebra is called for.

#### 2.2.1 Basic matrix operations

In matrix algebra a set of mathematical rules are given for operating on the basic elements real numbers, vectors, and matrices. In Mat1ab the type of each variable is determined when you first define it.

\[
\texttt{>> a=1} \\
\texttt{a = 1} \\
\texttt{>> b=2} \\
\texttt{b = 2} \\
\texttt{>> c=3} \\
\texttt{c = 3} \\
\texttt{>> y=[1;2;3]} \\
\texttt{y =} \\
\texttt{1} \\
\texttt{2} \\
\texttt{3} \\
\texttt{>> x=[1,2,3]} \\
\texttt{x =} \\
\texttt{1 2 3}
\]
A = [1 2 3; 4 5 6]

Observe that when filling in a vector or a matrix a space or a comma means a new number, a semicolon a new row. To suppress the printing of what you just defined, end the line with a semicolon:

A = [1,2,3,4;]
A = [1,2,3,4]

You can also use defined variables to define new variables, as long as the dimensions make sense. For example, given the above definitions:

B = [c x]
B =
3 1 2 3
C = [A;x]
C =
1 2 3
4 5 6
1 2 3
D = [A y]
error: number of rows must match
error: evaluating assignment expression near line 22, column 3

If the dimensioning is wrong, you get an error message, and the variable is not defined.

To see what is in a variable, tell Matlab to print the value by giving the name:

a = 1
A =
1 2 3
4 5 6

Note that Matlab is case-sensitive, both A and a are defined.

2.2.2 Arithmetic Matrix Operations.

We now get to the important parts of Matlab, namely, its built-in matrix arithmetic. Given the definitions above, let us add and subtract a few elements according to the rules of matrix algebra. We first show how to manipulate numbers and vectors:

a = 1
a = 1
b = 2
b = 2
a+b
ans = 3
>> x=[1 2 3 4]
x =  
   1   2   3   4
>> y=[4 3 2 1]
y =  
   4   3   2   1
>> x+y
ans =  
   5   5   5   5
>> y-x
ans =  
   3   1  -1  -3
>> a*x+b*y
ans =  
   9   8   7   6

similarly, for matrices:

A=[1 2 3; 4 5 6]
A =  
   1   2   3
   4   5   6
>> B=[6 5 4; 3 2 1]
B =  
   6   5   4
   3   2   1
>> A+B
ans =  
   7   7   7
   7   7   7
>> A-B
ans =  
  -5  -3  -1
    1   3   5
>> a*A+b*B
ans =  
  13  12  11
   10   9   8
> A*B'
ans =  
   28   10
   73  28

In linear algebra, you need to be aware that matrix multiplication is not element by element multiplication, it is a much more complex operation, where all possible vector combinations are multiplied with each other. When multiplying matrices, you need to be more careful about dimensions, but in terms of notation it is just like vector multiplication.

>> A=[1 2 3; 4 5 6]
A =  
   1   2   3
   4   5   6
>> B = [1 2; 3 4; 5 6]
B = 
1  2
3  4
5  6
>> A*B
ans =
22  28
49  64
>> B*A
ans =
   9  12  15
  19  26  33
  29  40  51

For these matrices, both \( AB \) and \( BA \) are defined operations, but note that the results are different, in fact, even the dimension of the product is different.

If we let \( B \) be a \( 2 \times 2 \) matrix, then multiplying \( AB \) is an error.

>> B=[1 2;3 4]
B =
1  2
3  4
>> A*B
error: nonconformant matrices (op1 is 2x3, op2 is 2x2)
>> B*A
ans =
   9  12  15
  19  26  33

Let us now discuss some standard matrix concepts.

The transpose of a matrix \( A \) is found in Matlab as \( A' \):

>> A
A =
1  2  3
4  5  6
>> A'
ans =
1  4
2  5
3  6

Two special matrices are the null and identity matrices:

>> null = zeros(3,3)
null =
0  0  0
0  0  0
0  0  0
>> ident = eye(3,3)
ident =
1  0  0
0  1  0

21
The rank of a matrix is the number of independent rows or columns in the matrix, and calculated as

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
\]

\[\text{rank}(A)\]

ans = 2

The inverse of a square matrix \(A\) is the matrix \(\text{inv}(A)\) such that \(A \times \text{inv}(A)\) equals the identity matrix, or in mathematical notation \(AA^{-1} = I\).

\[
\begin{bmatrix}
1 & 2 \\
1 & 4
\end{bmatrix}
\]

\[\text{inv}(D)\]

ans =

\[
\begin{bmatrix}
2.00000 & -1.00000 \\
-0.50000 & 0.50000
\end{bmatrix}
\]

\[D^{-1}\]

ans =

\[
\begin{bmatrix}
2.00000 & -1.00000 \\
-0.50000 & 0.50000
\end{bmatrix}
\]

To make sure that this is the inverse, multiply \(D\) and \(\text{inv}(D)\):

\[D \times \text{inv}(D)\]

ans =

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Determinant

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\]

\[\text{det}(B)\]

ans = -2

### 2.3 Solving linear equations

Consider the basic linear equation

\[Ax = b\]

This equation has a defined solution if the rank of \(A\) equals the rank of \([A|b]\). If \(A\) is nonsingular, we solve the linear equation by finding the unique solution

\[x = A^{-1}b\]
Consider the linear equation

\[3x_1 + 4x_2 = 5\]
\[4x_1 + 6x_2 = 8\]

Write this in matrix form by defining

\[A = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}\]
\[b = \begin{bmatrix} 5 \\ 8 \end{bmatrix}\]

Let us first check that this system is solvable

\[
\begin{align*}
\text{>> } & A = [3 \ 4; \ 4 \ 6] \\
& A = \\
& \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix} \\
\text{>> } & b = [5; 8] \\
& b = \\
& \begin{bmatrix} 5 \\ 8 \end{bmatrix} \\
\text{>> } & \text{rank}(A) \\
& \text{ans} = 2 \\
\text{>> } & \text{rank } ([A \ b]) \\
& \text{ans} = 2
\end{align*}
\]

Note how to create the augmented matrix \([A|b]\) by \([A \ b]\). The rank of the two is the same. Since \(A\) is square, we can calculate the solution as

\[
\begin{align*}
\text{>> } & \text{inverse}(A) \\
& \text{ans} = \\
& \begin{bmatrix} 3.0000 & -2.0000 \\ -2.0000 & 1.5000 \end{bmatrix} \\
\text{>> } & x = \text{inverse}(A) \cdot b \\
& x = \\
& \begin{bmatrix} -1 \\ 2 \end{bmatrix}
\end{align*}
\]

The solution to the system of equations is

\[x = \begin{bmatrix} -1 \\ 2 \end{bmatrix}\]

In this case we calculated the solution by finding the inverse. But you should be aware that solving the system of equations by calculation of the inverse is not the numerically most stable way of doing the calculation. Matlab has built in a direct linear matrix solver, which is invoked by the \textit{left division} operator

\[
\begin{align*}
\text{>> } & x = A\backslash b \\
& x = \\
& \begin{bmatrix} -1 \\ 2 \end{bmatrix}
\end{align*}
\]
This solves the system of equations directly, and it is usually the preferred way to do this operation, unless the inverse is needed for other purposes.

### 2.4 Element by element operations

When a command is prefixed with a period, it means the command applies to each element of a vector or matrix, not the vector or matrix.

For example, with the two vectors below, consider the difference in multiplying the two and doing an element by element multiplication:

```matlab
> x=[1 2 3 ]
x =
   1 2 3
> t=[1 2 3]
t =
   1 2 3
>> x*t'
anst = 14
>> x.*t
ans =
   1 4 9
```

Similarly, when taking the exponent of a matrix

```matlab
> A=[1 1;1 1]
A =
   1 1
   1 1
>> A^10
ans =
   512 512
   512 512
>> A.^10
ans =
   1 1
   1 1
```

### 2.5 Function definitions

### 2.6 m files

### 2.7 Flow control

### 2.8 Plotting

A very important use for a tool like Matlab is its ability to produce plots graphing mathematical relationships. For illustrating mathematical relations a two or three dimensional picture can be better than the thousands words of the old adage.
2.9 Libraries

2.10 References

You need the manual for your chosen package.
Chapter 3

The value of time

Contents

3.1 Present value ................................................................. 26
3.2 One interest rate with annual compounding ....................... 27
  3.2.1 Internal rate of return .................................................. 30
3.3 Continously compunded interest ........................................ 34
  3.3.1 Present value ............................................................ 35
3.4 Further readings ............................................................. 35

Finance as a field of study is sometimes somewhat flippantly said to deal with the value of two things: time and risk. While this is not the whole story, there is a deal of truth in it. These are the two issues which is always present. We start our discussion by ignoring risk and only considering the implications of the fact that anybody prefers to get something earlier rather than later, or the value of time.

3.1 Present value

The present value is the current value of a stream of future payments. Let $C_t$ be the cash flow at time $t$. Suppose we have $N$ future cash flows that occur at times $t_1, t_2, \ldots, t_N$.

To find the present value of these future cash flows one need a set of prices of future cash flows. Suppose $d_t$ is the price one would pay today for the right to receive one dollar at a future date $t$. Such a price is also called a discount factor. To complicate matters further such prices will differ depending on the riskiness of the future cash flows. For now we concentrate on one particular set of prices, the prices of riskless future cash flows. We will return to how one would adjust the prices for risky cash flows.

If one knows the set of prices for future claims of one dollar, $d_1, d_2, \ldots$, one would calculate the present value as the sum of the present values of the different elements.

$$PV = \sum_{i=1}^{N} d_{t_i} C_{t_i}$$
However, knowing this set of current prices for cash flows at all future dates is not always feasible, and some way has to be found to simplify the data need inherent in such general present value calculations.

### 3.2 One interest rate with annual compounding

The best known way to simplify the present value calculation is to rewrite the discount factors in terms of interest rates, or yields, through the relationship:

\[ d_t = \frac{1}{(1 + r_t)^t} \]

where \( r_t \) is the interest rate (usually termed the spot rate) relevant for a \( t \)-period investment. To further simplify this calculation one can impose that this interest rate \( r \) is constant for all periods. This is termed a flat term structure. We will in the next chapter relax this simplifying assumption. The prices for valuing the future payments \( d_t \) is calculated from this interest rate:

\[ d_t = \frac{1}{(1 + r)^t} \]

In this case one would calculate the present value of a stream of cash flows paid at discrete dates \( t = 1, 2, \ldots, N \) as

\[ PV = \sum_{t=1}^{N} \frac{c_t}{(1 + r)^t}. \]

The implementation of this calculation is shown in **C++ Code 3.1**.

```cpp
def cash_flow_pv_discrete(const vector<double>& cflow_times, const vector<double>& cflow_amounts, const double& r)
{
    double PV=0.0;
    for (int t=0; t<cflow_times.size(); t++)
    {
        PV += cflow_amounts[t]/pow(1.0+r,cflow_times[t]);
    }
    return PV;
}

C++ Code 3.1: Present value with discrete compounding
```

**Example**
An investment project has an investment cost of 100 today, and produces cash flows of 75 each of the next two years. What is the Net Present Value of the project?

Matlab program:

\[

c = [-100 75 75] \\
t = [0 1 2] \\
r = 0.1 \\
d = (1/(1+r))^{-t} \\
NPV = c' * d'
\]

Output from Matlab program:

\[
c = \\
-100 75 75 \\
t = \\
0 1 2 \\
r = 0.10000 \\
d = \\
1.00000 0.90909 0.82645 \\
NPV = 30.165
\]

C++ program:

```cpp
vector<double> cflows; cflows.push_back(-100.0); cflows.push_back(75); cflows.push_back(75); 
vector<double> times; times.push_back(0.0); times.push_back(1); times.push_back(2); 
double r = 0.1; 
cout << " Present value, 10 percent discretely compounded interest = " 
     << cash_flow_pv_discrete(times, cflows, r) << endl; 
```

Output from C++ program:

```
Present value, 10 percent discretely compounded interest = 30.1653
```

Given the assumption of a discrete, annual interest rate, there are a number of useful special cases of cash flows where one can calculate the present value in a simplified manner. Some of these are shown in the following exercises.

**Exercise 3.1.**

A perpetuity is a promise of a payment of a fixed amount \( X \) each period for the indefinite future. Suppose there is a fixed interest rate \( r \).

1. Show that the present value of this sequence of cash flows is calculated simply as

\[
PV = \sum_{t=1}^{\infty} \frac{X}{1+r} = \frac{X}{r}
\]

**Exercise 3.2.**

A growing perpetuity is again an infinite sequence of cashflows, where the payment the first year is \( X \) and each consequent payment grows by a constant rate \( g \), i.e, the time 2 payment is \( X(1+g) \), the time 3 payment is \( X(1+g)^2 \), and so on.

1. Show that the present value of this perpetuity simplifies to

\[
PV = \sum_{t=1}^{\infty} \frac{X(1+g)^{t-1}}{(1+r)^t} = \frac{X}{r - g}
\]
Exercise 3.3.

An annuity is a sequence of cashflows for a given number of years, say $T$ periods into the future. Consider an annuity paying a fixed amount $X$ each period. The interest rate is $r$.

1. Show that the present value of this sequence of cash flows can be simplified as

$$PV = \sum_{t=1}^{T} \frac{X}{(1+r)^t} = X \left[ \frac{1}{r} - \frac{1}{r(1+r)^T} \right]$$

Exercise 3.4.

An growing annuity is a sequence of cashflows for a given number of years, say $T$ periods into the future, where each payment grows by a given factor each year. Consider a $T$-period annuity that pays $X$ the first period. After that, the payments grows at a rate of $g$ per year, i.e. the second year the cash flow is $X(1+g)$, the third $X(1+g)^2$, and so on.

1. Show that the present value of this growing annuity can be simplified as

$$PV = \sum_{t=1}^{T} \frac{X(1+g)^{(t-1)}}{(1+r)^t} = X \left[ \frac{1}{r-g} - \left( \frac{1+g}{1+r} \right)^T \frac{1}{r-g} \right]$$

Exercise 3.5.

Rank the following cash flows in terms of present value. Use an interest rate of 5%.

1. A perpetuity with an annual payment of $100.
2. A growing perpetuity, where the first payment is $75, and each subsequent payment grows by 2%.
3. A 10-year annuity with an annual payment of $90.
4. A 10 year growing annuity, where the first payment is $85, and each subsequent payment grows by 5%.
3.2.1 **Internal rate of return.**

In addition to its role in simplifying present value calculations, the interest rate has some further use. The percentage return on an investment is a summary measure of the investment’s profitability. Saying that an investment earns 10% per year is a good way of summarizing the cash flows in a way that does not depend on the amount of the initial investment. The return is thus a relative measure of profitability. To estimate a return for a set of cash flows we calculate the *internal rate of return*. The internal rate of return for a set of cash flows is the interest rate that makes the present value of the cash flows equal to zero. When there is a uniquely defined internal rate of return we get a relative measure of the profitability of a set of cash flows, measured as a return, typically expressed as a percentage. Note some of the implicit assumptions made here. We assume that the same interest rate applies at all future dates (i.e. a flat term structure). The IRR method also assumes intermediate cash flows are reinvested at the internal rate of return.

Suppose the cash flows are \( C_0, C_1, C_2, \ldots, C_T \). Finding an internal rate of return is finding a solution \( y \) of the equation

\[
\sum_{t=1}^{T} \frac{C_t}{(1 + y)^t} - C_0 = 0
\]

Note that this is a polynomial equation, and as \( T \) becomes large, there in no way to find an explicit solution to the equation. It therefore needs to be solved numerically. For well behaved cash flows, where we know that there is one IRR, the method implemented in C++ Code 3.2 is suitable, it is an iterative process called bisection. It is an adaption of the bracketing approach discussed in (Press, Teukolsky, Vetterling, and Flannery, 1992, Chapter9),
```cpp
#include <cmath>
#include <algorithm>
#include <vector>
using namespace std;
#include "fin_recipes.h"

const double ERROR = -1e30;

double cash_flow_irr_discrete(const vector<double>& cflow_times,
const vector<double>& cflow_amounts) {
    // simple minded irr function. Will find one root (if it exists.)
    // adapted from routine in Numerical Recipes in C.
    if (cflow_times.size() != cflow_amounts.size()) return ERROR;
    const double ACCURACY = 1.0e-5;
    const int MAX_ITERATIONS = 50;
    double x1 = 0.0;
    double x2 = 0.2;

    // create an initial bracket, with a root somewhere between bot,top
    double f1 = cash_flow_pv_discrete(cflow_times, cflow_amounts, x1);
    double f2 = cash_flow_pv_discrete(cflow_times, cflow_amounts, x2);
    int i;
    for (i = 0; i < MAX_ITERATIONS; i++) {
        if ((f1 * f2) < 0.0) {
            //
            if (fabs(f1) < fabs(f2)) {
                f1 = cash_flow_pv_discrete(cflow_times, cflow_amounts, x1 += 1.6 * (x1 - x2));
            } else {
                f2 = cash_flow_pv_discrete(cflow_times, cflow_amounts, x2 += 1.6 * (x2 - x1));
            }
        } else {
            rtb = x2;
            dx = x2 - x1;
            };
        if (f2 * f1 > 0.0) {
            return ERROR;
        }
        double f = cash_flow_pv_discrete(cflow_times, cflow_amounts, x1);
        double rtb;
        double dx = 0;
        if (f < 0.0) {
            rtb = x1;
            dx = x2 - x1;
        } else {
            rtb = x2;
            dx = x1 - x2;
        }
    for (i = 0; i < MAX_ITERATIONS; i++) {
        dx *= 0.5;
        double x_mid = rtb + dx;
        double f_mid = cash_flow_pv_discrete(cflow_times, cflow_amounts, x_mid);
        if (fabs(f_mid) < 0.0) {
            rtb = x_mid;
        } else {
            rtb = x_mid;
        } if (fabs(dx) < ACCURACY) return x_mid;
    return ERROR; // error.
    }

C++ Code 3.2: Estimation of the internal rate of return
```
Example

We are considering an investment with the following cash flows at dates 0, 1 and 2:

\[ C_0 = -100, \quad C_1 = 10, \quad C_2 = 110 \]

1. The current interest rate (with discrete, annual compounding) is 5%. Determine the present value of the cash flows.

2. Find the internal rate of return of this sequence of cash flows.

Matlab program:

```matlab
C=[-100 10 110];
t=[0 1 2];
r = 0.05;
d=(1/(1+r)).^t;
NPV=C*d';
IRR = irr(C)
```

Output from Matlab program:

```
C =
   -100   10  110
   0   1   2
r = 0.050000
   d =
1.000000   0.95238   0.90703
NPV = 9.2971
IRR = 0.100000
```

C++ program:

```c++
vector<double> cflows; cflows.push_back(-100.0); cflows.push_back(10.0); cflows.push_back(110.0);
vector<double> times; times.push_back(0.0); times.push_back(1); times.push_back(2);
double r=0.05;
cout << " present value, 5 percent discretely compounded interest = " ;
cout << cash_flow_pv_discrete(times, cflows, r) << endl;
cout << " internal rate of return, discrete compounding = " ;
cout << cash_flow_irr_discrete(times, cflows) << endl;
```

Output from C++ program:

```
present value, 5 percent discretely compounded interest = 9.29705
internal rate of return, discrete compounding = 0.1
```
In addition to the above economic qualifications to interpretations of the internal rate of return, we also have to deal with technical problem stemming from the fact that any polynomial equation has potentially several solutions, some of which may be imaginary. By imaginary here we mean that we move away from the real line to the set of complex numbers. In economics we prefer the real solutions, complex interest rates are not something we have much intuition about... To see whether we are likely to have problems in identifying a single meaningful IRR, the code shown in code 3.3 implements a simple check. It is only a necessary condition for a unique IRR, not sufficient, so you may still have a well-defined IRR even if this returns false. The first test is just to count the number of sign changes in the cash flow. From Descartes rule we know that the number of real roots is one if there is only one sign change. If there is more than one change in the sign of cash flows, we can go further and check the aggregated cash flows (See Norstrom (1972)).

```cpp
#include <cmath>
#include <vector>
using namespace std;

inline int sgn(const double & r){ if (r>=0) {return 1;} else {return -1;});

bool cash_flow_unique_irr(const vector<double> & cflow_times,
                          const vector<double> & cflow_amounts) {
    int sign_changes=0; // first check Descartes rule
    for (int t=1;t<cflow_times.size();t++){
        if (sgn(cflow_amounts[t-1]) != sgn(cflow_amounts[t])) sign_changes++;
    }
    if (sign_changes==0) return false; // can not find any irr
    if (sign_changes==1) return true;

    double A = cflow_amounts[0]; // check the aggregate cash flows, due to Norstrom
    sign_changes=0;
    for (int t=1;t<cflow_times.size();t++){
        if (sgn(A) != sgn(A+cflow_amounts[t])) sign_changes++;
    }
    if (sign_changes<=1) return true;
    return false;
}
```

**C++ Code 3.3: Test for uniqueness of IRR**

A better way to gain an understanding for the relationship between the interest rate and the present value is simply to plot the present value as a function of the interest rate. The following picture illustrates the method for two different cash flows. Note that the set of cash flows on the right has two possible interest rates that sets the present value equal to zero.

\[ C_0 = -100, C_1 = 10, C_2 = 100 \]

\[ C_0 = -100, C_1 = 201, C_2 = -100 \]
Exercise 3.6.

An alternative way of estimating the IRR is to use an external subroutine that finds the root of a polynomial equation. Search for a suitable general subroutine for root finding and replace the IRR estimation with a call to this subroutine.

3.3 Continuously compounded interest

Such discrete compounding as we have just discussed is not the only alternative way to approximate the discount factor. The discretely compounded case assumes that interest is added at discrete points in time (hence the name). However, an alternative assumption is to assume that interest is added continuously. If compounding is continuous, and \( r \) is the interest rate, one would calculate the current price \( d_t \) of receiving one dollar at a future date \( t \) as

\[
d_t = e^{-rt},
\]

Formula 3.1 summarizes some rules for translating between continously compounded and discretly compounded interest rates.

\[
\begin{align*}
  r &= n \ln \left(1 + \frac{r_n}{n}\right) \\
  r_n &= n \left(e^r - 1\right) \\
  \text{Future value} &= e^{rt} \\
  \text{Present value} &= e^{-rt}
\end{align*}
\]

Notation: \( r_n \): interest rate with discrete compounding, \( n \): compounding periods per year. \( r \): interest rate with continuous compounding, \( t \): time to maturity.

**Formula 3.1:** Translating between discrete and continuous compounding

Example

1. Given a 15% interest rate with monthly compounding, calculate the equivalent interest rate with continuous compounding.

2. Given a 12% interest rate with continuous compounding, find the equivalent interest rate with quarterly compounding.

Carrying out the calculations:

1. \( r_{12} = 15\% \), \( r = 12 \ln \left(1 + \frac{0.15}{12}\right) = 14.91\% \)

2. \( r = 12\% \), \( r_4 = n \left(e^r - 1\right) = 4 \left(e^{0.12/4} - 1\right) = 12.18\% \)

Using Matlab to do the calculations:

Matlab program:

```
r = 12 * log( 1+0.15/12)
r4 = 4 * ( exp(0.12/4)-1 )
```

Output from Matlab program:

```
r = 0.14907
r4 = 0.12182
```
3.3.1 Present value

Applying this to a set of cash flows at future dates $t_1, t_2, \ldots, t_n$, we get the following present value calculation:

$$PV = \sum_{i=1}^{n} e^{-r t_i} C_i$$

This calculation is implemented in C++ Code 3.4.

```cpp
#include <cmath>
#include <vector>
using namespace std;

double cash_flow_pv( const vector<double>& cflow_times,
    const vector<double>& cflow_amounts,
    const double& r) {
    double PV = 0.0;
    for (int t = 0; t < cflow_times.size(); t++) {
        PV += cflow_amounts[t] * exp(-r * cflow_times[t]);
    }
    return PV;
}
```

C++ Code 3.4: Present value calculation with continuously compounded interest

In much of what follows we will work with the case of continuously compounded interest. There is a number of reasons why, but a prime reason is actually that it is easier to use continuously compounded interest than discretely compounded, because it is easier to deal with uneven time periods. Discretely compounded interest is easy to use with evenly spaced cash flows (such as annual cash flows), but harder otherwise.

3.4 Further readings

The material in this chapter is covered in most standard textbooks of corporate finance (e.g. Brealey, Myers, and Allen (2010) or Ross, Westerfield, and Jaffe (2009)) and investments (e.g. Bodie, Kane, and Marcus (2007), Haugen (2001) or Sharpe, Alexander, and Bailey (1999)).
Chapter 4

Bond Pricing with a flat term structure

Contents

4.1 Flat term structure with discrete, annual compounding
4.1.1 Bond Price
4.1.2 Yield to maturity
4.1.3 Duration
4.1.4 Measuring bond sensitivity to interest rate changes
4.2 Continuously compounded interest
4.3 Further readings

In this section we use the present value framework of the previous chapter to price bonds and other fixed income securities. What distinguishes bonds is that the future payments are set when the security is issued. The simplest, and most typical bond, is a fixed interest, constant maturity bond with no default risk. There is however a large number of alternative contractual features of bonds. The bond could for example be an annuity bond, paying a fixed amount each period. For such a bond the principal amount outstanding is paid gradually during the life of the bond. The interest rate the bond pays need not be fixed, it could be a floating rate, the interest rate paid could be a function of some market rate. Many bonds are issued by corporations, and in such cases there is a risk that the company issued the bond defaults, and the bond does not pay the complete promised amount. Another thing that makes bond pricing difficult in practice, is that interest rates tend to change over time.

We start by assuming that all the promised payments are certain. Then the bond current price $B_0$ is found as the present value of these payments. The first step of pricing is to use the terms of the bond to find the promised payments. We start by considering a fixed interest bond with no default risk. Such bonds are typically bonds issued by governments. The bond is a promise to pay a face value $F$ at the maturity date $T$ periods from now. Each period the bond pays a fixed percentage amount of the face value as coupon $C$. The cash flows from the bond thus look as follows.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon</td>
<td>$C$</td>
<td>$C$</td>
<td>$C$</td>
<td>$C$</td>
<td></td>
<td>$C$</td>
</tr>
<tr>
<td>Face value</td>
<td></td>
<td>$F$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Total cash flows $C_1 = C$, $C_2 = C$, $\cdots$, $C_T = C + F$.

In general a bond price is found as the present value

$$B_0 = d_1 C_1 + d_2 C_2 + \cdots + d_T C_T = \sum_{t=1}^{T} d_t C_t$$

where $d_t$ is the discount factor, or the time 0 price of a payment of 1 at time $t$. To fully specify the
problem it is necessary to find all discount factors \( d_t \). In this chapter we will work with a specially simple specification of the term structure, namely that it is flat, and specified by the interest rate \( r \).

### 4.1 Flat term structure with discrete, annual compounding

This is the simplest possible specification of a term structure,

\[
d_t = \left(\frac{1}{1 + r}\right)^t = \frac{1}{(1 + r)^t}
\]

#### 4.1.1 Bond Price

The current bond price \( B_0 \) is the present value of the cash flows from the bond

\[
B_0 = \sum_{t=1}^{T} \left( \frac{1}{(1 + r)^t} \right) C_t = \sum_{t=1}^{T} \frac{C_t}{(1 + r)^t} \tag{4.1}
\]

If we continue with the example of a standard fixed interest bond, where \( C_t = C \) when \( t < T \) and \( C_T = C + F \), and show how the bond price will be calculated using Matlab.

**Example**

A 3 year bond with a face value of $100 makes annual coupon payments of 10%. The current interest rate (with annual compounding) is 9%.

1. Determine the current bond price.

The current bond price: \( B_0 = \frac{10}{(1+0.09)^1} + \frac{10}{(1+0.09)^2} + \frac{110}{(1+0.09)^3} = 102.531 \).

Here are the calculations:

Matlab program:

```matlab
C = [10,10,110];
t = 1:3;
r = 0.09;
d = (1./((1+r).^t));
B = d * C;
```

Output from Matlab program:

- \( C = \begin{bmatrix} 10 & 10 & 110 \end{bmatrix} \)
- \( t = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \)
- \( r = 0.09 \)
- \( d = \begin{bmatrix} 0.91743 & 0.84168 & 0.77218 \end{bmatrix} \)
- \( B = 102.531 \)

C++ program:

```cpp
vector<double> cflows; cflows.push_back(10); cflows.push_back(10); cflows.push_back(110);
vector<double> times; times.push_back(1); times.push_back(2); times.push_back(3);
double r=0.09;
cout " bonds price = "<< bonds_price_discrete(times, cflows, r) << endl;
```

Output from C++ program:

- \( \text{bonds price} = 102.531 \)
The general code in C++ for calculating the bond price with discrete annual compounding is shown in C++ Code 4.1.

```cpp
#include <cmath>
#include <vector>

using namespace std;

double bonds_price_discrete(const vector<double>& times,
                           const vector<double>& cashflows,
                           const double& r) {
  double p = 0;
  for (int i = 0; i < times.size(); i++) {
    p += cashflows[i] / (pow((1 + r), times[i]));
  }
  return p;
}
```

C++ Code 4.1: Bond price calculation with discrete, annual compounding.

### 4.1.2 Yield to maturity

Since bonds are issued in terms of interest rate, it is also useful to find an interest rate number that summarizes the terms of the bond. The obvious way of doing that is asking the question: What is the internal rate of return on the investment of buying the bond now and keeping the bond to maturity? The answer to that question is the yield to maturity of a bond. The yield to maturity is the interest rate that makes the present value of the future coupon payments equal to the current bond price, that is, for a known price $B_0$, the yield is the solution $y$ to the equation

$$B_0 = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

(4.2)

This calculation therefore has the same qualifications as discussed earlier calculating IRR, it supposes reinvestment of coupon at the bond yield (the IRR).

There is much less likelihood we’ll have technical problems with multiple solutions when doing this yield estimation for bonds, since the structure of cash flows is such that there exist only one solution to the equation. The algorithm for finding a bonds yield to maturity shown in C++ Code 4.2 is thus simple bisection. We know that the bond yield is above zero and set zero as a lower bound on the bond yield. We then find an upper bound on the yield by increasing the interest rate until the bond price with this interest rate is negative. We then bisect the interval between the upper and lower until we are “close enough.” C++ Code 4.2 implements this idea.
```cpp
#include <cmath>
using namespace std;
#include "fin_recipes.h"

double bonds_yield_to_maturity_discrete( const vector<double>& times,
                                        const vector<double>& cashflows,
                                        const double& bondprice ) {

    const double ACCURACY = 1e-5;
    const int MAX_ITERATIONS = 200;
    double bot=0, top=1.0;
    while (bonds_price_discrete(times, cashflows, top) > bondprice) { top = top*2; };
    double r = 0.5 * (top+bot);
    for (int i=0;i<MAX_ITERATIONS;i++) {
        double diff = bonds_price_discrete(times, cashflows,r) - bondprice;
        if (fabs(diff)<ACCURACY) return r;
        if (diff>0.0) { bot=r; } else { top=r; }
        r = 0.5 * (top+bot);
    }
    return r;
}
```

**C++ Code 4.2:** Bond yield calculation with discrete, annual compounding
**Example**

A 3 year bond with a face value of $100 makes annual coupon payments of 10%. The current interest rate (with annual compounding) is 9%.

1. Find the bond's current price.
2. Find the bond's yield to maturity.

### Matlab program:

```matlab
C = [ 10 10 110 ];
t = 1:3;
r = 0.09;
B = C * (1./((1+r).^t))';
y = irr([-B C])
```

**Output from Matlab program:**

- B = 102.53
- y = 0.090000

### C++ program:

```cpp
vector<double> cflows; cflows.push_back(10); cflows.push_back(10); cflows.push_back(110);
vector<double> times; times.push_back(1); times.push_back(2); times.push_back(3);
double r = 0.09;
double B = bonds_price_discrete(times, cflows, r);
cout << " Bond price, 9 percent discretely compounded interest = " << B << endl;
cout << " bond yield to maturity = " << bonds_yield_to_maturity_discrete(times, cflows, B) << endl;
```

**Output from C++ program:**

- Bond price, 9 percent discretely compounded interest = 102.531
- bond yield to maturity = 0.09
4.1.3 Duration

When holding a bond one would like to know how sensitive the value of the bond is to changes in economic environment. The most relevant piece of the economic environment is the current interest rate. An important component of such calculation is the duration of a bond. The duration of a bond should be interpreted as the weighted average maturity of the bond, and is calculated as

\[
\text{Duration} = \frac{\sum_t t \cdot \frac{C_t}{(1+r)^t}}{\text{Bond Price}} \tag{4.3}
\]

where \( C_t \) is the cash flow in period \( t \), and \( r \) the interest rate. Using the bond price calculated in equation 4.1 we calculate duration as

\[
D = \frac{\sum_t \frac{t \cdot C_t}{(1+r)^t}}{\sum_t \frac{C_t}{(1+r)^t}} \tag{4.4}
\]

which is shown in C++ Code 4.3.

```cpp
#include <cmath>
#include <vector>
using namespace std;

double bonds_duration_discrete(const vector<double>& times,
                               const vector<double>& cashflows,
                               const double& r) {
    double B=0;
    double D=0;
    for (int i=0;i<times.size();i++)
    {
        D += times[i] * cashflows[i] / pow(1+r,times[i]);
        B += cashflows[i] / pow(1+r,times[i]);
    }
    return D/B;
}
```

C++ Code 4.3: Bond duration using discrete, annual compounding and a flat term structure

An alternative approach to calculating duration is calculate the yield to maturity \( y \) for the bond, and use that in estimating the bond price. This is called Macaulay Duration. First one calculates \( y \), the yield to maturity, from

\[
\text{Bond price} = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}
\]

and then use this \( y \) in the duration calculation:

\[
\text{Macaulay duration} = \frac{\sum_t t \cdot \frac{C_t}{(1+y)^t}}{\sum_t \frac{C_t}{(1+y)^t}} \tag{4.5}
\]

C++ Code 4.4 implements this calculation.

Note though, that in the present case, with a flat term structure, these should produce the same number. If the bond is priced correctly, the yield to maturity must equal the current interest rate. If \( r = y \) the two calculations in equations (4.4) and (4.5) obviously produces the same number.

Example

A 3 year bond with a face value of $100 makes annual coupon payments of 10%. The current interest rate (with annual compounding) is 9%.
```cpp
#include "fin_recipes.h"

double bonds_duration_macaulay_discrete(const vector<double>& times,
                                       const vector<double>& cashflows,
                                       const double& bond_price) {
    double y = bonds_yield_to_maturity_discrete(times, cashflows, bond_price);
    return bonds_duration_discrete(times, cashflows, y); // use YTM in duration calculation
}

C++ Code 4.4: Calculating the Macaulay duration of a bond

1. Determine the current bond price.
2. Calculate the duration using the current interest rate.
3. Calculate the duration using the MaCaulay definition.

Need to calculate the following:
The current bond price: \( B_0 = \frac{10}{(1+0.09)^1} + \frac{10}{(1+0.09)^2} + \frac{110}{(1+0.09)^3} = 102.531 \).
The bond’s duration: \( D = \frac{1}{102.531} (\frac{1}{1.09} + \frac{2}{1.09^2} + \frac{3}{1.09^3}) = 2.74 \).

Matlab program:
```matlab
C = [10, 10, 110];
t = 1:3;
r = 0.09;
B = C .* (1 ./ (1+r).^t);
D = (1/B) .* C .* (1 ./ (1+r).^t);
y = irr([-B C])
DM = (1/B) .* C .* (1 ./ (1+y).^t)
```

Output from Matlab program:

\( B = 102.53 \)
\( D = 2.7390 \)
\( y = 0.090000 \)
\( DM = 2.7390 \)

C++ program:
```cpp
vector<double> cflows; cflows.push_back(10); cflows.push_back(10); cflows.push_back(110);
vector<double> times; times.push_back(1); times.push_back(2); times.push_back(3);
double r=0.09;
double B = bonds_price_discrete(times, cflows, r);
cout << " bonds price = " << B << endl;
cout << " bond duration = " << bonds_duration_discrete(times, cflows, r) << endl;
cout << " bond macaulay = " << bonds_duration_macaulay_discrete(times, cflows, B) << endl;
```

Output from C++ program:

\( \text{bonds price} = 102.531 \)
\( \text{bond duration} = 2.73895 \)
\( \text{bond macaulay} = 2.73895 \)

42
4.1.4 Measuring bond sensitivity to interest rate changes

Now, the reason for why we say that we can measure the sensitivity of a bond price using duration. To a first approximation, $\frac{\Delta P}{P_0}$, the change in the bond price for a small change in the interest rate $\Delta r$, can be calculated:

$$\frac{\Delta P}{P_0} \approx - \frac{D}{1 + r} \Delta r$$

where $D$ is the bond’s duration. For simplicity one often calculates the term in front of the $\Delta y$ in the above, $\frac{D}{1+y}$ directly and terms it the bond’s modified duration.

Modified Duration $= D^* = \frac{D}{1 + r}$

The sensitivity calculation is then

$$\frac{\Delta P}{P_0} \approx - D^* \Delta r$$

The modified duration is also written in term’s of the bond’s yield to maturity $y$, and is then

$$D^* = \frac{D}{1 + y}$$

C++ Code 4.5 shows this calculation.

```cpp
#include <vector>
using namespace std;
#include "fin_recipes.h"

double bonds_duration_modified_discrete (const vector<double>& times,
const vector<double>& cashflows,
const double& bond_price){
    double y = bonds_yield_to_maturity_discrete(times, cashflows, bond_price);
    double D = bonds_duration_discrete(times, cashflows, y);
    return D/(1+y);
};
```

C++ Code 4.5: Modified duration

Approximating bond price changes using duration is illustrated in the following figure.
The modified duration measures the angle of the tangent at the current bond yield. Approximating the change in bond price with duration is thus only a first order approximation. To improve on this approximation we also need to account for the curvature in the relationship between bond price and interest rate. To quantify this curvature we calculate the \textit{convexity} of a bond.

\[
\text{Convexity} = Cx = \frac{1}{B_0} \frac{1}{(1 + r)^2} \sum_{t=1}^{T} \frac{t C_t}{(1 + r)^t}
\] (4.6)

This calculation is implemented in C++ Code 4.6. To improve on the estimate of how the bond price change when the interest rates changes you will then calculate

\[
\frac{\Delta B_0}{B_0} \approx -D^* \Delta y + \frac{1}{2} Cx (\Delta y)^2
\]

Formula 4.1 summarizes the above calculations.

<table>
<thead>
<tr>
<th>Bond Price ((B_0))</th>
<th>Modified duration</th>
<th>Convexity ((Cx))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B_0 = \sum_{t=1}^{T} \frac{C_t}{(1 + r)^t})</td>
<td>(D^* = \frac{D}{1 + y})</td>
<td>(Cx = \frac{1}{B_0} \frac{1}{(1 + r)^2} \sum_{t=1}^{T} \frac{t C_t}{(1 + r)^t})</td>
</tr>
<tr>
<td>Yield to maturity (y) solves</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(B_0 = \sum_{t=1}^{T} \frac{C_t}{(1 + y)^t})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Duration ((D))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(D = \frac{1}{B_0} \sum_{t=1}^{T} \frac{t C_t}{(1 + r)^t})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Macaulay duration</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(D = \frac{1}{B_0} \sum_{t=1}^{T} \frac{t C_t}{(1 + y)^t})</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(C_t\): Cash flow at time \(t\), \(r\): Interest rate, \(y\): Bond yield to maturity, \(B_0\): Current bond price. Bond pays coupon at evenly spaced dates \(t = 1, 2, 3 \ldots, T\).

\textbf{Formula 4.1}: Bond pricing formulas with a flat term structure and discrete, annual compounding
Example

A 3 year bond with a face value of $100 makes annual coupon payments of 10%. The current interest rate (with annual compounding) is 9%.

1. Determine the current bond price.

2. Suppose the interest rate changes to 10%, determine the new price of the bond by direct calculation.

3. Use duration to estimate the new price and compare it to the correct price.

4. Use convexity to improve on the estimate using duration only.

Need to calculate the following:

The current bond price:

\[ B_0 = \frac{10}{1+0.09} + \frac{10}{(1+0.09)^2} + \frac{110}{(1+0.09)^3} = 102.531 \]

The bond’s duration:

\[ D = \frac{1}{102.531} \left( \frac{10}{1.09} + \frac{2\times10}{1.09^2} + \frac{3\times110}{1.09^3} \right) = 2.74 \]

The modified duration:

\[ D^* = \frac{D}{1+r} = \frac{2.74}{1+0.09} = 2.51 \]

The convexity:

\[ C_x = \frac{1}{(1+0.09)^2} \left( \frac{(1+1)\times10}{1.09} + \frac{(2+2)\times10}{1.09^2} + \frac{(3+3)\times110}{1.09^3} \right) = 8.93 \]

Here are the calculations:

Matlab program:

```matlab
C = [10, 10, 110];
t = 1:3;
r = 0.09;
B = C * (1./(1+r).^t);
D = (1/B).*t.*C * (1./(1+r).^t);
Cx = (1/(1+r).^2) * (1/B).*t.*C * (1./(1+r).^t);
newB = C * (1./(1+0.1).^t);
```

Output from Matlab program:

- \( B = 102.53 \)
- \( D = 2.7390 \)
- \( C_x = 6.6272 \)
- \( newB = 100.000 \)

C++ program:

```cpp
vector<double> cflows; cflows.push_back(10); cflows.push_back(10); cflows.push_back(110);
vector<double> times; times.push_back(1); times.push_back(2); times.push_back(3);
double r = 0.09;
double B = bonds_price_discrete(times, cflows, r);
cout << " bonds price = " << B << endl;
cout << " bond duration = " << bonds_duration_discrete(times, cflows, r) << endl;
cout << " bond duration modified = " << bonds_duration_modified_discrete(times, cflows, B) << endl;
cout << " bond convexity = " << bonds_convexity_discrete(times, cflows, r) << endl;
cout << " new bond price = " << bonds_price_discrete(times, cflows, 0.1) << endl;
```

Output from C++ program:

- Bonds price = 102.531
- Bond duration = 2.73895
- Bond duration modified = 2.5128
- Bond convexity = 8.93248
- New bond price = 100
Using these numbers to answer the questions, let us see what happens when the interest rate increases to 10%. This means the bond will be selling at par, equal to 100, which can be confirmed with direct computation:

\[ B_0 = \frac{10}{(1 + 0.1)^1} + \frac{10}{(1 + 0.1)^2} + \frac{110}{(1 + 0.1)^3} = 100 \]

Using duration to estimate the change in the bond price for a unit change in the interest rate:

\[ \frac{\Delta B_0}{B_0} = -D \Delta y = -2.51 \cdot 0.01 = -0.0251 \]

Using this duration based number to estimate the new bond price.

\[ \tilde{B} = B_0 + \left( \frac{\Delta B_0}{B_0} \right) B_0 = 102.531 - 0.0251 \cdot 102.531 = 99.957 \]

Additionally using convexity in estimating the change in the bond price:

\[ \frac{\Delta B_0}{B_0} = -D \Delta y + \frac{1}{2} Cx \Delta y^2 = -2.51 \cdot 0.01 + \frac{1}{2} 8.93(0.01)^2 = -0.0251 + 0.00044 = -0.02465 \]

\[ \tilde{B} = 102.531 \left( 1 + \left( \frac{\Delta B_0}{B_0} \right) B_0 \right) = 102.531(1 - 0.02465) = 100.0036 \]

Exercise 4.1.

**Perpetual duration** [4]

The term structure is flat with annual compounding. Consider the pricing of a perpetual bond. Let \( C \) be the per period cash flow

\[ B_0 = \sum_{t=1}^{\infty} \frac{C}{(1 + r)^t} = \frac{C}{r} \]

1. Determine the first derivative of the price with respect to the interest rate.
2. Find the duration of the bond.

Exercise 4.2.

[5]

Consider an equally weighted portfolio of two bonds, A and B. Bond A is a zero coupon bond with 1 year to maturity. Bond B is a zero coupon bond with 3 years to maturity. Both bonds have face values of 100. The current interest rate is 5%.

1. Determine the bond prices.
2. Your portfolio is currently worth 2000. Find the number of each bond invested.
3. Determine the duration of the portfolio.
4. Determine the convexity of your position.
4.2 Continuously compounded interest

We will go over the same concepts as covered in the previous section on bond pricing. There are certain subtle differences in the calculations. Formula 4.2 corresponds to the earlier summary in formula 4.1.

<table>
<thead>
<tr>
<th>Bond Price $B_0$:</th>
<th>Convexity $C_x$:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_0 = \sum_i e^{-rt_i} C_{t_i}$</td>
<td>$C_x = \frac{1}{B_0} \sum_i C_{t_i} t_i^2 e^{-rt_i}$</td>
</tr>
</tbody>
</table>

Yield to maturity $y$ solves:

<table>
<thead>
<tr>
<th>Duration $D$:</th>
<th>Approximating bond price changes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D = \frac{1}{B_0} \sum_i t_i C_{t_i} e^{-rt_i}$</td>
<td>$\frac{\Delta B_0}{B_0} \approx -D \Delta y$</td>
</tr>
</tbody>
</table>

Macaulay duration

| $D = \frac{1}{B_0} \sum_i t_i C_{t_i} e^{-yt_i}$ | $\frac{\Delta B_0}{B_0} \approx -D \Delta y + \frac{1}{2} C_x \times (\Delta y)^2$ |

Bond paying cash flows $C_{t_1}, C_{t_2}, \ldots$ at times $t_1, t_2, \ldots$. Notation: $B_0$: current bond price. $e$: natural exponent.

**Formula 4.2:** Bond pricing formulas with continuously compounded interest and a flat term structure

Some important differences is worth pointing out. When using continuously compounded interest, one does not need the concept of *modified duration*. In the continuously compounded case one uses the calculated duration directly to approximate bond changes, as seen in the formulas describing the approximation of bond price changes. Note also the difference in the *convexity* calculation, one does not divide by $(1 + y)^2$ in the continuously compounded formula, as was done in the discrete case.


```cpp
#include <cmath>
#include <vector>
using namespace std;

double bonds_price(const vector<double>& cashflow_times, const vector<double>& cashflows, const double& r) {
    double p = 0;
    for (int i=0;i<cashflow_times.size();i++) {
        p += exp(-r*cashflow_times[i])*cashflows[i];
    }
    return p;
}
```

C++ Code 4.7: Bond price calculation with continuously compounded interest and a flat term structure
```cpp
#include <cmath>
#include <vector>
#include "fin_recipes.h"

using namespace std;

double bonds_duration(const vector<double>& cashflow_times,
                       const vector<double>& cashflows,
                       const double& r) {
    double S=0;
    double D1=0;
    for (int i=0;i<cashflow_times.size();i++){
        S += cashflows[i] * exp(-r*cashflow_times[i]);
        D1 += cashflow_times[i] * cashflows[i] * exp(-r*cashflow_times[i]);
    }
    return D1 / S;
}

c++ Code 4.8: Bond duration calculation with continuously compounded interest and a flat term structure

```}

```cpp
#include "fin_recipes.h"

double bonds_duration_macaulay(const vector<double>& cashflow_times,
                                const vector<double>& cashflows,
                                const double& bond_price) {
    double y = bonds_yield_to_maturity(cashflow_times, cashflows, bond_price);
    return bonds_duration(cashflow_times, cashflows, y); // use YTM in duration
}

c++ Code 4.9: Calculating the Macaulay duration of a bond with continuously compounded interest and a flat term structure

```}

```cpp
#include <cmath>
#include "fin_recipes.h"

double bonds_convexity(const vector<double>& times,
                        const vector<double>& cashflows,
                        const double& r ) {
    double C=0;
    for (int i=0;i<times.size();i++){
        C += cashflows[i] * pow(times[i],2) * exp(-r*times[i]);
    }
    double B=bonds_price(times, cashflows,r);
    return C/B;
}

c++ Code 4.10: Bond convexity calculation with continuously compounded interest and a flat term structure

```
Example

A 3 year bond with a face value of $100 makes annual coupon payments of 10%. The current interest rate (with continuous compounding) is 9%.

1. Calculate the bond’s price, yield to maturity, duration and convexity.

2. Suppose the interest rate falls to 8%. Estimate the new bond price using duration, and compare with the actual bond price using the correct interest rate.

Calculations:

Matlab program:

```
C = [10, 10, 110]
t = 1:3
r = 0.09
d = (1./(1+r).^t)
B = C * d'
D = 1/B*C.*t*d'
Dadj = D/(1+r)
r = 0.1
actualB = C*(1./(1+r).^t)
```

Output from Matlab program:

```
C =
10 10 110
t =
1 2 3
r = 0.090000
d =
0.91743 0.84168 0.77218
B = 102.53
D = 2.7390
Dadj = 2.5128
r = 0.10000
actualB = 100.000
```

C++ program:

```
vector<double> cflows; cflows.push_back(10); cflows.push_back(10); cflows.push_back(110);
vector<double> times; times.push_back(1); times.push_back(2); times.push_back(3);
double r = 0.09;
double B = bonds_price(times, cflows, r);
cout << " bonds price = " << B << endl;
cout << " bond duration = " << bonds_duration(times, cflows, r) << endl;
cout << " bond convexity = " << bonds_convexity(times, cflows, r) << endl;
cout << " new bond price = " << bonds_price(times, cflows, 0.08);
```

Output from C++ program:

```
bonds price = 101.464
bond duration = 2.73753
bond convexity = 7.86779
new bond price = 104.282
```

Exercise 4.3.
The term structure is flat, and compounding is continuous. Consider two definitions of duration, the “usual” definition $D = \frac{1}{B_0} \sum_i t_i C_i e^{-r t_i}$ and the Macaulay definition: $D = \frac{1}{B_0} \sum_i t_i C_i e^{-y t_i}$, where $B_0$ is the current bond price, $C_i$ is the coupon payment at date $t_i$, $r$ is the current interest rate and $y$ is the bond’s yield to maturity.

1. Show that these two definitions will produce the same number if the bond is correctly priced.

### 4.3 Further readings

The material in this chapter is covered in most standard textbooks on investments (e.g. Bodie et al. (2007), Haugen (2001) or Sharpe et al. (1999)). More details can be found in textbooks on fixed income, such as Sundaresan (2001).
Chapter 5

The term structure of interest rates and an object lesson

Contents

5.1 The interchangeability of discount factors, spot interest rates and forward interest rates . . . . . 52
5.2 The term structure as an object . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 55
  5.2.1 Base class . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 55
  5.2.2 Flat term structure. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 57
5.3 Using the currently observed term structure. . . . . . . . . . . . . . . . . . . . . . . . . . . . 58
  5.3.1 Linear Interpolation. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 59
  5.3.2 Interpolated term structure class. . . . . . . . . . . . . . . . . . . . . . . . . . . 61
5.4 Bond calculations with a general term structure and continous compounding . . . . . . . . . . . 64

In this chapter we expand on the analysis of the previous chapter by relaxing the “one interest rate” assumption used there and allow the spot rates to change as you change the time you are discounting over.

Recall that we said that the present value of a set of cash flows is calculated as

\[ PV = \sum_{t=1}^{N} d_t C_t \]

To make this applicable to cash flows received at any future date \( t \) we potentially need an infinite number of discount factors \( d_t \). This is not feasible, so some lower dimensional way needs to be found to approximate \( d_t \), but with more flexibility than the extremely strong assumption that there is one fixed interest rate \( r \), and that the discount factor for any time \( t \) is calculated as either \( d_t = 1/(1+r)^t \) (discrete compounding), or \( d_t = e^{-rt} \) (continuous compounding), which we used in the previous chapter.
In this chapter we first show that this approximation of the discount factors can be done in either terms of discount factors directly, interest rates, or forward rates. Either of these are useful ways of formulating a term structure, and either of them can be used, since there are one to one transformations between either of these three. We then go on to demonstrate how a feature of C++, the ability to create an abstract datatype as an object, or class, is very useful for the particular application of defining and using a term structure. It is in fact this particular application, to create a term structure class, which really illustrates the power of C++, and why you want to use an object oriented language instead of classical languages like FORTRAN and C, or matrix languages like Gauss or Matlab for many financial calculations.

5.1 The interchangeability of discount factors, spot interest rates and forward interest rates

The term structure can be specified in terms of either discount factors, spot interest rates or forward interest rates. A discount factor is the current price for a future (time $t$) payment of one dollar. To find the current value $PV$ of a cash flow $C_t$, we calculate $PV = d_t C_t$. This discount factor can also be specified in terms of interest rates, where we let $r_t$ be the relevant interest rate (spot rate) for discounting a $t$-period cashflow. Then we know that the present value $PV = e^{-r_t t} C_t$. Since these two methods of calculating the present value must be consistent, $PV = d_t C_t = e^{-r_t t} C_t$ and hence $d_t = e^{-r_t t}$

Note that this equation calculates $d_t$ given $r_t$. Rearranging this equation we find the spot rate $r_t$ in terms of discount factors

$$r_t = \frac{-\ln(d_t)}{t}$$

An alternative concept that is very useful is a forward interest rate, the yield on borrowing at some future date $t_1$ and repaying it at a later date $t_2$. Let $f_{t_1,t_2}$ be this interest rate. If we invest one dollar today, at the current spot rate spot rate till period $t_1$ and the forward rate for the period from $t_1$ to $t_2$ (which is what you would have to do to make an actual investment), you would get the following future value

$$FV = e^{r_{t_1} t_1} e^{f_{t_1,t_2} (t_2-t_1)}$$

The present value of this forward value using the time $t_2$ discount factor has to equal one:

$$d_{t_2} FV = 1$$

These considerations are enough to calculate the relevant transforms. The forward rate for borrowing at time $t_1$ for delivery at time $t_2$ is calculated as

$$f_{t_1,t_2} = \frac{-\ln \left( \frac{d_{t_2}}{d_{t_1}} \right)}{t_2 - t_1} = \frac{\ln \left( \frac{d_{t_2}}{d_{t_1}} \right)}{t_2 - t_1}$$

The forward rate can also be calculated directly from yields as

$$f_{t_1,t_2} = r_{t_2} \frac{t_2}{t_2 - t_1} - r_{t_1} \frac{t_1}{t_2 - t_1}$$
\[ d_t = e^{-r_t t} \]
\[ r_t = \frac{-\ln(d_t)}{t} \]
\[ f_{t_1, t_2} = \frac{\ln\left( \frac{d_{t_1}}{d_{t_2}} \right)}{t_2 - t_1} \]
\[ f_{t_1, t_2} = \frac{t_2}{t_2 - t_1} - \frac{r_{t_1}}{t_2 - t_1} \]

Notation: \( d_t \) discount factor for payment at time \( t \), \( r_t \): spot rate applying to cash flows at time \( t \). \( f_{t_1, t_2} \) forward rate between time \( t_1 \) and \( t_2 \), i.e. the interest rate you would agree on today on the future transactions.

```cpp
#include <cmath>
using namespace std;

double term_structure_yield_from_discount_factor(const double& d_t, const double& t) {
    return (-log(d_t)/t);
}
double term_structure_discount_factor_from_yield(const double& r, const double& t) {
    return exp(-r*t);
};
double term_structure_forward_rate_from_discount_factors(const double& d_t1, const double& d_t2,
                                                        const double& time) {
    return (log (d_t1/d_t2))/time;
};
double term_structure_forward_rate_from_yields(const double& r_t1, const double& r_t2,
                                             const double& t1, const double& t2) {
    return r_t2*t2/(t2-t1)-r_t1*t1/(t2-t1);
};
```

C++ Code 5.1: Term structure transformations

C++ Code 5.1 shows the implementation of these transformations.

Example
You are given the one period spot rate \( r_1 = 5\% \) and the two period discount factor \( d_2 = 0.9 \). Calculate the two period spot rate and the forward rate from 1 to 2.
C++ program:

double t1=1; double r_t1=0.05; double d_t1=term_structure_discount_factor_from_yield(r_t1,t1);
cout << " a " << t1 << " period spot rate of " << r_t1
     " corresponds to a discount factor of " << d_t1 << endl;
double t2=2; double d_t2 = 0.9;
double r_t2 = term_structure_yield_from_discount_factor(d_t2,t2);
cout << " a " << t2 << " period discount factor of " << d_t2
     " corresponds to a spot rate of " << r_t2 << endl;
cout << " the forward rate between " << t1 << " and " << t2
     " is " << term_structure_forward_rate_from_discount_factors(d_t1,d_t2,t2-t1)
     " using discount factors " << endl;
cout << " and is " << term_structure_forward_rate_from_yields(r_t1,r_t2,t1,t2)
     " using yields " << endl;

Output from C++ program:

a 1 period spot rate of 0.05 corresponds to a discount factor of 0.951229
a 2 period discount factor of 0.9 corresponds to a spot rate of 0.0526803
the forward rate between 1 and 2 is 0.0553605 using discount factors
and is 0.0553605 using yields
5.2 The term structure as an object

From the previous we see that the term structure can be described in terms of discount factors, spot rates or forward rates, but that does not help us in getting round the dimensionality problem. If we think in terms of discount factors, for a complete specification of the current term structure one needs an infinite number of discount factors \( \{d_t\}_{t \in \mathbb{R}_+} \). It is perhaps easier to think about this set of discount factors as a function \( d(t) \), that, given a nonnegative time \( t \), returns the discount factor. Since we have established that there are three equivalent ways of defining a term structure, discount factors, spot rates and forward rates, we can therefore describe a term structure as a collection of three different functions that offer different views of the same underlying object.

A term structure is an abstract object that to the user should provide

- discount factors \( d \) (prices of zero coupon bonds).
- spot rates \( r \) (yields of zero coupon bonds).
- forward rates \( f \)

for any future maturity \( t \). The user of a term structure will not need to know how the term structure is implemented, all that is needed is an interface that specifies the above three functions.

This is tailor made for being implemented as a C++ class. A class in C++ terms is a collection of data structures and functions that operate on these data structures. In the present context it is a way of specifying the three functions.

\[ r(t) \]

\[ d(t) \]

\[ f(t) \]

5.2.1 Base class

Header File 5.1 shows how we describe the generic term structure as a C++ class.

```cpp
#ifndef _TERM_STRUCTURE_CLASS_H_
#define _TERM_STRUCTURE_CLASS_H_

class term_structure_class {
public:
  virtual double r(const double& t) const; // yield on zero coupon bond
  virtual double d(const double& t) const; // discount factor/price of zero coupon bond
  virtual double f(const double& t1, const double& t2) const; // forward rate
  virtual ~term_structure_class();
};
#endif
```

Header file 5.1: Header file describing the term_structure base class

The code for these functions uses algorithms that are described earlier in this chapter for transforming between various views of the term structure. The term structure class merely provide a convenient interface to these algorithms. The code is shown in C++ Code 5.2

Note that the definitions of calculations are circular. Any given specific type of term structure has to over-ride at least one of the functions \( r \) (yield), \( d \) (discount factor) or \( f \) (forward rate).

We next consider two examples of specific term structures.
#include "fin_recipes.h"

term_structure_class::term_structure_class();

double term_structure_class::f(const double& t1, const double& t2) const{
    double d1 = d(t1);
    double d2 = d(t2);
    return term_structure_forward_rate_from_discount_factors(d1,d2,t2-t1);
};

double term_structure_class::r(const double& t) const{
    return term_structure_yield_from_discount_factor(d(t),t);
};

double term_structure_class::d(const double& t) const {
    return term_structure_discount_factor_from_yield(r(t),t);
};

C++ Code 5.2: Default code for transformations between discount factors, spot rates and forward rates in
a term structure class
5.2.2 Flat term structure.

The flat term structure overrides the `yield` member function of the base class.
The only piece of data this type of term structure needs is an interest rate.

```
#define _TERM_STRUCTURE_CLASS_FLAT_
#include _TERM_STRUCTURE_CLASS_FLAT_

#include "term_structure_class.h"
class term_structure_class_flat : public term_structure_class {
private:
  double R_;  // interest rate
public:
  term_structure_class_flat(const double& r);
  virtual ~term_structure_class_flat();
  virtual double r(const double& t) const;
  void set_int_rate(const double& r);
};
#endif
```

Header file 5.2: Header file for term structure class using a flat term structure

```
#include "fin_recipes.h"
term_structure_class_flat::term_structure_class_flat(const double& r){ R_ = r; };
term_structure_class_flat::~term_structure_class_flat();
double term_structure_class_flat::r(const double& T) const { if (T>=0) return R_; return 0; };
void term_structure_class_flat::set_int_rate(const double& r) { R_ = r; };
```

C++ Code 5.3: Implementing term structure class using a flat term structure

Example
The term structure is flat with \( r = 5\% \). Determine the discount factors for years 1 and 2 and the forward rate between 1 and 2.

```
C++ program:

term_structure_class_flat ts(0.05);
double t1=1;
cout << "discount factor t1 = " << t1 << ":" << ts.d(t1) << endl;
double t2=2;
cout << "discount factor t2 = " << t2 << ":" << ts.d(t2) << endl;
cout << "spot rate t = " " << t1 << ":" << ts.r(t1) << endl;
cout << "spot rate t = " " << t2 << ":" << ts.r(t2) << endl;
cout << "forward rate from t1= " " t1 " to t2= " << t2 << " :" << ts.f(t1,t2) << endl;
```

Output from C++ program:

discount factor t1 = 1:0.951229
discount factor t2 = 2:0.904837
spot rate t = 1:0.05
spot rate t = 2:0.05
forward rate from t1= 1 to t2= 2:0.05
```
5.3 Using the currently observed term structure.

A first step to a general term structure is to use market data on bond prices and interest rates to infer discount factors. The simplest way of presenting this problem is in terms of linear algebra. Suppose we have three bonds with cash flows at times 1, 2 and 3.

\[
\begin{align*}
B_1 &= d_1 C_{11} + d_2 C_{12} + d_2 C_{13} \\
B_2 &= d_1 C_{21} + d_2 C_{22} + d_2 C_{23} \\
B_3 &= d_1 C_{31} + d_2 C_{32} + d_2 C_{33}
\end{align*}
\]

Writing this in terms of matrices

\[
\begin{bmatrix}
B_1 \\
B_2 \\
B_3
\end{bmatrix} =
\begin{bmatrix}
d_1 \\
d_2 \\
d_3
\end{bmatrix} \cdot
\begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix}
\]

Solving for the discount factors

\[
\begin{bmatrix}
d_1 \\
d_2 \\
d_3
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix}^{-1} \cdot
\begin{bmatrix}
B_1 \\
B_2 \\
B_3
\end{bmatrix}
\]

or

\[
d = C^{-1} B
\]

using the obvious matrix definitions

\[
B =
\begin{bmatrix}
B_1 \\
B_2 \\
B_3
\end{bmatrix},
\quad d =
\begin{bmatrix}
d_1 \\
d_2 \\
d_3
\end{bmatrix},
\quad C =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix}
\]

Example

The following set of bond and bond prices is observed:

<table>
<thead>
<tr>
<th>Bond</th>
<th>Time (in years)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>98</td>
</tr>
<tr>
<td>2</td>
<td>96</td>
</tr>
<tr>
<td>3</td>
<td>92</td>
</tr>
<tr>
<td>4</td>
<td>118</td>
</tr>
<tr>
<td>5</td>
<td>109</td>
</tr>
<tr>
<td></td>
<td>112</td>
</tr>
</tbody>
</table>

The bonds are treasury securit{ies, which can be viewed as nominally riskless.

1. For what maturities is it possible to infer discount factors?
2. Determine the implicit discount factors in the market.
3. What is the interest rates implied in this set of discount rates?

Here we can find the discount factors for maturities of 0.5, 1, 1.5, 2, 3 and 4.
Matlab program:

\[
\begin{align*}
C &= \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 10 & 10 & 10 & 110 & 0 & 0 \\ 8 & 8 & 8 & 8 & 108 & 0 \\ 9 & 9 & 9 & 9 & 9 & 109 \end{bmatrix} \\
B &= \begin{bmatrix} 96 & 94 & 92 & 118 & 109 & 112 \end{bmatrix} \\
d &= B \cdot \text{inv}(C)' \\
t &= [0.5 \ 1.5 \ 2 \ 3 \ 4] \\
r &= d \cdot (-1/t - 1)
\end{align*}
\]

Output from Matlab program:

\[
\begin{align*}
C &= \\
&= \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 10 & 10 & 10 & 110 & 0 & 0 \\ 8 & 8 & 8 & 8 & 108 & 0 \\ 9 & 9 & 9 & 9 & 9 & 109 \end{bmatrix} \\
B &= \\
&= \begin{bmatrix} 96 & 94 & 92 & 118 & 109 & 112 \end{bmatrix} \\
d &= \begin{bmatrix} 0.96000 & 0.94000 & 0.92000 & 0.81636 & 0.73990 & 0.66618 \end{bmatrix} \\
t &= \begin{bmatrix} 0.50000 & 1.00000 & 1.50000 & 2.00000 & 3.00000 & 4.00000 \end{bmatrix} \\
r &= \begin{bmatrix} 0.085069 & 0.063830 & 0.057162 & 0.106772 & 0.105628 & 0.106884 \end{bmatrix}
\end{align*}
\]

To just use today's term structure, we need to take the observations of discount factors \(d_t\) observed in the market and use these to generate a term structure. The simplest possible way of doing this is to linearly interpolate the currently observable discount factors.

### 5.3.1 Linear Interpolation.

If we are given a set of discount factors \(d_t\) for various maturities, the simplest way to construct a term structure is by straightforward linear interpolation between the observations we have to find an intermediate time. For many purposes this is "good enough." This interpolation can be on either zero coupon yields, discount factors or forward rates, we illustrate the case of linear interpolation of spot (zero coupon) rates.

**Computer algorithm, linear interpolation of yields.** Note that the algorithm assumes the yields are ordered in increasing order of time to maturity.

**Example**

You observe the following term structure of spot rates

<table>
<thead>
<tr>
<th>Time</th>
<th>(r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
</tr>
<tr>
<td>5</td>
<td>0.4</td>
</tr>
<tr>
<td>10</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Interpolate spot rates (zero rates) at times 0.1, 0.5, 1, 3, 5 and 10.
double term_structure_yield_linearly_interpolated(const double& time,  
    const vector<double>& obs_times,  
    const vector<double>& obs_yields) {
  // assume the yields are in increasing time to maturity order.
  int no_obs = int(obs_times.size());
  if (no_obs<1) return 0;
  double t_min = obs_times[0];
  if (time <= t_min) return obs_yields[0]; // earlier than lowest obs.
  double t_max = obs_times[no_obs-1];
  if (time >= t_max) return obs_yields[no_obs-1]; // later than latest obs
  int t=1; // find which two observations we are between
  while ((t<no_obs) && (time>obs_times[t])) { ++t; }
  double lambda = (obs_times[t-1]-time)/(obs_times[t]-obs_times[t-1]);
  // by ordering assumption, time is between t-1,t
  double r = obs_yields[t-1] * lambda + obs_yields[t] * (1.0-lambda);
  return r;
}

C++ code 5.4: Interpolated term structure from spot rates

C++ program:

```cpp
vector<double> times;
vector<double> yields;
times.push_back(0.1); times.push_back(0.5); times.push_back(1);
yields.push_back(0.1); yields.push_back(0.2); yields.push_back(0.3);
times.push_back(5); times.push_back(10);
yields.push_back(0.4); yields.push_back(0.5);
cout << " yields at times: " << endl;
cout << " t=0.1 " << term_structure_yield_linearly_interpolated(0.1,times,yields) << endl;
cout << " t=0.5 " << term_structure_yield_linearly_interpolated(0.5,times,yields) << endl;
cout << " t=1 " << term_structure_yield_linearly_interpolated(1,times,yields) << endl;
cout << " t=3 " << term_structure_yield_linearly_interpolated(3,times,yields) << endl;
cout << " t=5 " << term_structure_yield_linearly_interpolated(5,times,yields) << endl;
cout << " t=10 " << term_structure_yield_linearly_interpolated(10,times,yields) << endl;
```

Output from C++ program:

```
yields at times:
t=0.1 0.1
t=0.5 0.2
t=1 0.3
t=3 0.35
t=5 0.4
t=10 0.5
```
5.3.2 Interpolated term structure class.

The interpolated term structure implemented here uses a set of observations of yields as a basis, and for observations in between observations will interpolate between the two closest. The following only provides implementations of calculation of the yield, for the other two rely on the base class code. As shown in Header File 5.3 and C++ Code 5.5, there is some more book-keeping involved here, need to have code that stores observations of times and yields.

Header file 5.3: Header file describing a term structure class using linear interpolation between spot rates. 

```c++
#ifndef TERM_STRUCTURE_CLASS_INTERPOLATED
#define TERM_STRUCTURE_CLASS_INTERPOLATED

#include "term_structure_class.h"
#include <vector>

using namespace std;

class term_structure_class_interpolated : public term_structure_class {

private:
    vector<double> times_; // use to keep a list of yields
    vector<double> yields_;  
    void clear();

public:
    term_structure_class_interpolated();
    term_structure_class_interpolated(const vector<double>& times, const vector<double>& yields);
    virtual "term_structure_class_interpolated();
    term_structure_class_interpolated(const term_structure_class_interpolated&);
    term_structure_class_interpolated operator= (const term_structure_class_interpolated&);

    int no_observations() const { return times_.size(); }
    virtual double r(const double& T) const;
    void set_interpolated_observations(vector<double>& times, vector<double>& yields);
};
#endif
```
#include "fin_recipes.h"

void term_structure_class_interpolated::clear()
{
    times_.erase(times_.begin(), times_.end());
    yields_.erase(yields_.begin(), yields_.end());
}

term_structure_class_interpolated::term_structure_class_interpolated()
    : term_structure_class_interpolated()
{
    clear();
}

term_structure_class_interpolated::term_structure_class_interpolated(const vector<double>& in_times,
                        const vector<double>& in_yields)
{
    clear();
    if (in_times.size()!=in_yields.size()) return;
    times_ = vector<double>(in_times.size());
    yields_ = vector<double>(in_yields.size());
    for (int i=0;i<in_times.size();i++)
    {
        times_[i]=in_times[i];
        yields_[i]=in_yields[i];
    }
}

term_structure_class_interpolated::term_structure_class_interpolated(const term_structure_class_interpolated& term)
{
    times_ = vector<double>(term.no_observations());
    yields_ = vector<double>(term.no_observations());
    for (int i=0;i<term.no_observations();i++)
    {
        times_[i] = term.times_[i];
        yields_[i] = term.yields_[i];
    }
}

term_structure_class_interpolated::operator= (const term_structure_class_interpolated& term)
{
    times_ = vector<double>(term.no_observations());
    yields_ = vector<double>(term.no_observations());
    for (int i=0;i<term.no_observations();i++)
    {
        times_[i] = term.times_[i];
        yields_[i] = term.yields_[i];
    }
    return (*this);
}

double term_structure_class_interpolated::r(const double& T) const
{
    return term_structure_yield_linearly_interpolated(T, times_, yields_);
}

void term_structure_class_interpolated::set_interpolated_observations(vector<double>& in_times,
                        vector<double>& in_yields)
{
    clear();
    if (in_times.size()!=in_yields.size()) return;
    times_ = vector<double>(in_times.size());
    yields_ = vector<double>(in_yields.size());
    for (int i=0;i<in_times.size();i++)
    {
        times_[i]=in_times[i];
        yields_[i]=in_yields[i];
    }
}
Determine discount factors and spot rates at times 1 and 2, and forward rate between 1 and 2.

C++ program:

```cpp
vector<double> times; times.push_back(0.1);
vector<double> spotrates; spotrates.push_back(0.05);
times.push_back(1); times.push_back(5);
spotrates.push_back(0.07); spotrates.push_back(0.08);
term_structure_class_interpolated ts(times, spotrates);
double t1=1;
cout << "discount factor t1 = " << t1 << ":" << ts.d(t1) << endl;
double t2=2;
cout << "discount factor t2 = " << t2 << ":" << ts.d(t2) << endl;
cout << "spot rate t = " << t1 << ":" << ts.r(t1) << endl;
cout << "spot rate t = " << t2 << ":" << ts.r(t2) << endl;
cout << "forward rate from t1= " << t1 << " to t2= " << t2 << " ":" << ts.f(t1, t2) << endl;
```

Output from C++ program:

```
discount factor t1 = 1:0.932394
discount factor t2 = 2:0.865022
spot rate t = 1:0.07
spot rate t = 2:0.0725
forward rate from t1= 1 to t2= 2:0.075
```
5.4 Bond calculations with a general term structure and continuous compounding

Coupon bond paying coupons at dates $t_1, t_2, \ldots$

Bond Price $B_0$:

$$B_0 = \sum_i d_i C_{t_i} = \sum_i e^{-r_{t_i} t_i} C_{t_i}$$

Duration $D$:

$$D = \frac{1}{B_0} \sum_i t_i d_i C_{t_i}$$

Duration $D$:

$$D = \frac{1}{B_0} \sum_i t_i e^{-r_{t_i} t_i} C_{t_i}$$

Duration $D$:

$$D = \frac{1}{B_0} \sum_i t_i e^{-y_{t_i} t_i} C_{t_i}$$

Yield to maturity $y$ solves:

$$B_0 = \sum_i C_i e^{-y_{t_i}}$$

Convexity $C_x$:

$$C_x = \frac{1}{B_0} \sum_i t_i^2 d_i C_{t_i}$$

Convexity $C_x$:

$$C_x = \frac{1}{B_0} \sum_i t_i^2 e^{-r_{t_i} t_i} C_{t_i}$$

Convexity $C_x$:

$$C_x = \frac{1}{B_0} \sum_i t_i^2 e^{-y_{t_i} t_i} C_{t_i}$$

Formula 5.1: Bond pricing with a continuously compounded term structure

C++ Code 5.6 and C++ Code 5.7 illustrate how one would calculate bond prices and duration if one has a term structure class.

```
#include "fin_recipes.h"

double bonds_price(const vector<double>& cashflow_times,
                    const vector<double>& cashflows,
                    const term_structure_class& d) {
    double p = 0;
    for (unsigned i=0;i<cashflow_times.size();i++) {
        p += d.d(cashflow_times[i])*cashflows[i];
    }
    return p;
}
```

C++ Code 5.6: Pricing a bond with a term structure class
#include "fin_recipes.h"

double bonds_duration(const vector<double>& cashflow_times,
                        const vector<double>& cashflow_amounts,
                        const term_structure_class& d) {
    double S=0;
    double D1=0;
    for (unsigned i=0;i<cashflow_times.size();i++){
        S += cashflow_amounts[i] * d.d(cashflow_times[i]);
        D1 += cashflow_times[i] * cashflow_amounts[i] * d.d(cashflow_times[i]);
    }
    return D1/S;
}

C++ Code 5.7: Calculating a bonds duration with a term structure class

#include <cmath>

double bonds_convexity(const vector<double>& cashflow_times,
                        const vector<double>& cashflow_amounts,
                        const term_structure_class& d) {
    double B=0;
    double Cx=0;
    for (unsigned i=0;i<cashflow_times.size();i++){
        B += cashflow_amounts[i] * d.d(cashflow_times[i]);
        Cx += pow(cashflow_times[i],2) * cashflow_amounts[i] * d.d(cashflow_times[i]);
    }
    return Cx/B;
}

C++ Code 5.8: Calculating a bonds convexity with a term structure class
Example
The term structure is flat with $r = 10\%$ continously compounded interest. Calculate price, duration, and convexity of a 10%, 2 year bond.

C++ program:

```cpp
vector<double> times; times.push_back(1); times.push_back(2);
vector<double> cashflows; cashflows.push_back(10); cashflows.push_back(110);
term_structure_class_flat tsflat(0.1);
cout << " price = " << bonds_price(times, cashflows, tsflat) << endl;
cout << " duration = " << bonds_duration(times, cashflows, tsflat) << endl;
cout << " convexity = " << bonds_convexity(times, cashflows, tsflat) << endl;
```

Output from C++ program:

```
price = 99.1088
duration = 1.9087
convexity = 3.72611
```

References  Shiller (1990) is a good reference on the use of the term structure.
Chapter 6

The Mean Variance Frontier

Contents

6.1 Setup ................................................. 67
6.2 The minimum variance frontier ..................... 69
6.3 Calculation of frontier portfolios .................. 69
6.4 The global minimum variance portfolio .......... 70
6.5 Efficient portfolios .................................. 72
6.6 The zero beta portfolio ................................ 73
6.7 Allowing for a riskless asset ......................... 73
6.8 Efficient sets with risk free assets ................. 74
6.9 Short-sale constraints ............................... 75
6.10 The Sharpe Ratio ..................................... 75
6.11 Equilibrium: CAPM .................................. 76
  6.11.1 Treynor ........................................ 76
  6.11.2 Jensen .......................................... 76
6.12 Working with Mean Variance and CAPM .......... 76
6.13 Mean variance analysis using matrix libraries ... 77

We now discuss a classical topic in finance, mean variance analysis. This leads to ways of accounting for
the riskiness of cashflows.

Mean variance analysis concerns investors choices between portfolios of risky assets, and how an investor
chooses portfolio weights. Let \( r_p \) be a portfolio return. We assume that investors preferences over
portfolios \( p \) satisfy a mean variance utility representation, \( u(p) = u(E[r_p], \sigma(r_p)) \), with utility increasing
in expected return \( (\partial u/\partial E[r_p] > 0) \) and decreasing in variance \( (\partial u/\partial \sigma(r_p) < 0) \). In this part we
consider the representation of the portfolio opportunity set of such decision makers. There are a number
of useful properties of this opportunity set which follows purely from the mathematical formulation of
the optimization problem. It is these properties we focus on here.

6.1 Setup

We assume there exists \( n \geq 2 \) risky securities, with expected returns \( e \)

\[
e = \begin{bmatrix}
E[r_1] \\
E[r_2] \\
\vdots \\
E[r_n]
\end{bmatrix}
\]
and covariance matrix $V$:

$$V = \begin{bmatrix}
\sigma(r_1, r_1) & \sigma(r_1, r_2) & \cdots \\
\sigma(r_2, r_1) & \sigma(r_2, r_2) & \cdots \\
\vdots & \vdots & \ddots \\
\sigma(r_n, r_1) & \sigma(r_n, r_2) & \cdots & \sigma(r_n, r_n)
\end{bmatrix}$$

The covariance matrix $V$ is assumed to be invertible.

A portfolio $p$ is defined by a set of weights $w$ invested in the risky assets.

$$w = \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\vdots \\
\omega_n
\end{bmatrix},$$

where $\omega_i$ is the fraction of the investors wealth invested in asset $i$. Note that the weights sum to one.

The expected return on a portfolio is calculated as

$$E[r_p] = w' e$$

and the variance of the portfolio is

$$\sigma^2(r_p) = w' V w$$

Example

An investor can invest in three assets with expected returns and variances as specified in the following table.

<table>
<thead>
<tr>
<th>t</th>
<th>Asset</th>
<th>$E[r]$</th>
<th>$\sigma^2(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10%</td>
<td>0.20</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>11.5%</td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>8%</td>
<td>0.15</td>
<td></td>
</tr>
</tbody>
</table>

The three assets are independent (uncorrelated).

1. Determine the expected return and standard deviation of an equally weighted portfolio of the three assets.

Matlab program:

```matlab
e=[0.1 0.11 0.08];
V=[ 0.2 0 0; 0 0.1 0; 0 0 0.15];
w=1/3*[1 1 1];
er=e'*w;
sigma=sqrt(w*V*w');
```

Output from Matlab program:

```
e =
 0.100000 0.110000 0.080000
V =
 0.20000 0.00000 0.00000
 0.00000 0.10000 0.00000
 0.00000 0.00000 0.15000
w =
 0.33333 0.33333 0.33333
er = 0.096667
sigma = 0.22361```
6.2 The minimum variance frontier

A portfolio is a frontier portfolio if it minimizes the variance for a given expected return, that is, a frontier portfolio $p$ solves

$$w_p = \arg\min_w \frac{1}{2} w' V w$$

subject to:

$$w'e = E[r_p]$$

$$w'1 = 1$$

The set of all frontier portfolios is called the minimum variance frontier.

6.3 Calculation of frontier portfolios

**Proposition 1** If the matrix $V$ is full rank, and there are no restrictions on shortsales, the weights $w_p$ for a frontier portfolio $p$ with mean $E[r_p]$ can be found as

$$w_p = g + hE[r_p]$$

where

$$g = \frac{1}{D} (B1' - Ae') V^{-1}$$

$$h = \frac{1}{D} (Ce' - A1') V^{-1}$$

$$A = 1' V^{-1} e$$

$$B = e' V^{-1} e$$

$$C = 1' V^{-1} 1$$

$$A = \begin{bmatrix} B & A \\ A & C \end{bmatrix}$$

$$D = BC - A^2 = |A|$$

**Proof**

Any minimum variance portfolio solves the program

$$w_p = \arg\min_w \frac{1}{2} w' V w$$

subject to

$$w'e = E[r_p]$$

$$w'1 = 1$$

Set up the Lagrangian corresponding to this problem

$$L(w, \lambda, \gamma|e, V) = \frac{1}{2} w' V w - \lambda (E[r_p] - w'e) - \gamma(1 - w'1)$$
Differentiate

$$\frac{\partial L}{\partial w} = w' \mathbf{V} - \lambda e' - \gamma \mathbf{1}' = 0$$

$$\frac{\partial L}{\partial \lambda} = E[r_p] - w' e = 0$$

$$\frac{\partial L}{\partial \gamma} = 1 - w' \mathbf{1} = 0$$

Rewrite conditions above as (note that this requires the invertibility of $\mathbf{V}$.

$$w' = \lambda e' \mathbf{V}^{-1} - \gamma \mathbf{1}' \mathbf{V}^{-1}$$  \hspace{1cm} (6.1)

$$w' e = E[r_p]$$  \hspace{1cm} (6.2)

$$w' \mathbf{1} = 1$$  \hspace{1cm} (6.3)

Post-multiply equation (6.1) with $e$ and recognise the expression for $E[r_p]$ in the second equation

$$w' e = E[r_p] = \lambda e' \mathbf{V}^{-1} e + \gamma \mathbf{1}' \mathbf{V}^{-1} e$$

Similarly post-multiply equation (6.1) with $\mathbf{1}$ and recognise the expression for $1$ in the third equation

$$w' \mathbf{1} = 1 = \lambda e' \mathbf{V}^{-1} \mathbf{1} + \gamma \mathbf{1}' \mathbf{V}^{-1} \mathbf{1}$$

With the definitions of $A$, $B$, $C$ and $D$ above, this becomes the following system of equations

$$\begin{cases} E[r_p] = \lambda B + \gamma A \\ 1 = \lambda A + \gamma C \end{cases}$$

Solving for $\lambda$ and $\gamma$, get

$$\gamma = \frac{B - AE[r_p]}{D}$$

$$\lambda = \frac{CE[r_p] - A}{D}$$

Plug in expressions for $\lambda$ and $\gamma$ into equation (6.1) above, and get

$$w' = \frac{1}{D} \left( B \mathbf{1}' - Ae' \right) \mathbf{V}^{-1} + \frac{1}{D} \left( C e' - A \mathbf{1}' \right) \mathbf{V}^{-1} E[r_p] = g + h E[r_p]$$

The portfolio defined by weights $g$ is a portfolio with expected return 0. The portfolio defined by weights $(g + h)$ is a portfolio with expected return 1. This implies the useful property that $g \mathbf{1}' = 1$, and $h \mathbf{1}' = 0$.

**Example**

An investor can invest in three assets with expected returns and variances as specified in the following table.

<table>
<thead>
<tr>
<th>$t$</th>
<th>Asset</th>
<th>$E[r]$</th>
<th>$\sigma^2(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>10%</td>
<td>0.20</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>11.5%</td>
<td>0.10</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>8%</td>
<td>0.15</td>
</tr>
</tbody>
</table>

The three assets are independent (uncorrelated).

1. What are the weights of the minimum variance portfolio with mean 9%?
Matlab program:

```matlab
e=[0.1 0.11 0.08]';
V=[ 0.2 0 0 0.1 ; 0 0.1 0 ; 0 0 0.15 ];
r=0.09
n = length(e)
a = ones(1,n)*inv(V)*e
b = e*inv(V)*e
c = ones(1,n)*inv(V)*ones(n,1)
A = [b a a c]
d = det(A)
g = 1/d*(b*ones(1,n) - a*e')*inv(V)
h = h = 1/d*(c*e' - a*ones(1,n))*inv(V)
w=g+h*r
```

Output from Matlab program:

```plaintext
e =
 0.10000
 0.11000
 0.08000
V =
 0.20000  0.00000  0.00000
 0.00000  0.10000  0.00000
 0.00000  0.00000  0.15000
r = 0.090000
n = 3
a = 2.1333
b = 0.21367
c = 21.667
A =
 0.21367  2.13333
 2.13333  21.66667
d = 0.078333
g =
 0.021277 -2.680851  3.659574
h =
 2.1277  31.9149 -34.0426
w =
 0.21277  0.19149  0.59574
```

This calculation is put into a Matlab function in Matlab Code 6.1.

```matlab
function w = min_variance_portfolio(e,V,r)
n = length(e);
a = ones(1,n)*inv(V)*e;
b = e*inv(V)*e;
c = ones(1,n)*inv(V)*ones(n,1);
A = [b a a c];
d = det(A);
g = 1/d*(b*ones(1,n) - a*e')*inv(V);
h = h = 1/d*(c*e' - a*ones(1,n))*inv(V);
w = w;g+h*r;
end
```

Matlab Code 6.1: Calculation of minimum variance portfolio for given return
Calculating the weights of the minimum variance portfolio given an interest rate

\[ w_p = g + h E[r_p] \]

where

\[ g = \frac{1}{D} (B1' - Ae') V^{-1} \]
\[ h = \frac{1}{D} (Ce' - A1') V^{-1} \]
\[ A = 1' V^{-1} e \]
\[ B = e' V^{-1} e \]
\[ C = 1' V^{-1} I \]
\[ D = BC - A^2 = \]

Notation: \( r_p \) desired portfolio returns, \( V \): covariance matrix of asset returns, \( e \): vector of expected asset returns.

6.4 The global minimum variance portfolio

The portfolio that minimizes variance regardless of expected return is called the global minimum variance portfolio. Let \( \text{mup} \) be the global minimum variance portfolio.

**Proposition 2 (Global Minimum Variance Portfolio)** The global minimum variance portfolio has weights

\[ w'_{\text{mup}} = (1' V^{-1} I)^{-1} 1' V^{-1} = \frac{1}{C} 1' V^{-1}, \]

expected return \( E[r_{\text{mup}}] = A \) and variance \( \text{var}(r_{\text{mup}}) = \frac{1}{\sigma} \).

6.5 Efficient portfolios

Portfolios on the minimum variance frontier with expected returns higher than or equal to \( E[r_{\text{mup}}] \) are called efficient portfolios.
6.6 The zero beta portfolio

Proposition 3  For any portfolio $p$ on the frontier, there is a frontier portfolio $zc(p)$ satisfying
\[ \text{cov}(r_{zc(p)}, r_p) = 0. \]
This portfolio is called the zero beta portfolio relative to $p$. The zero beta portfolio $zb(p)$ has return
\[ E[r_{zc(p)}] = \frac{A}{C} - \frac{D}{E[r_p] - \frac{A}{C}} \]

Note that if $p$ is an efficient portfolio on the mean variance frontier then $zc(p)$ is inefficient. Conversely, if $p$ is inefficient $zc(p)$ is efficient.

6.7 Allowing for a riskless asset.

Suppose have $N$ risky assets with weights $w$ and one riskless assets with return $r_f$.
Intuitively, the return on a portfolio with a mix of risky and risky assets can be written as
\[ E[r_p] = \text{weight in risky} \times \text{return risky} + \text{weight riskless} \times r_f \]
which in vector form is:
\[ E[r_p] = w' e + (1 - w'1) r_f \]

Proposition 4  An efficient portfolio in the presence of a riskless asset has the weights
\[ w_p = \mathbf{V}^{-1}(e - 1 r_f) \frac{E[r_p] - r_f}{H} \]
where
\[ H = (e - 1 r_f)' \mathbf{V}^{-1}(e - 1 r_f) \]
The variance of the efficient portfolio is

\[ \sigma^2(r_p) = \frac{(E[r_p] - r_f)^2}{H} \]

Note that standard deviation is a linear function of \( E[r_p] \). The efficient set is a line in mean-standard deviation space.

### 6.8 Efficient sets with risk free assets.

Suppose \( r_f < \frac{A}{C} \). Then the efficient set is the line from \((0, r_f)\) through tangency on the efficient set of risky assets.

Suppose \( r_f > \frac{A}{C} \). Then the efficient set is the two half-lines starting from \((0, r_f)\).

If \( r_f = \frac{A}{C} \), the weight in the risk free asset is one. The risky portfolio is an zero investment portfolio. The efficient set consists of two asymptotes toward the efficient set of risky assets.
6.9 Short-sale constraints

So far the analysis has put no restrictions on the set of weights \( w_p \) that defines the minimum variance frontier. For practical applications, existence of negative weights is problematic, since this involves selling securities short.

This has led to the investigation of restricted mean variance frontiers, where the weights are constrained to be non-negative.

**Definition 1** A short sale restricted minimum variance portfolio \( p \) solves

\[
\begin{align*}
    w_p &= \arg \min_w \frac{1}{2} w' V w \\
    \text{subject to} & \\
    w' e &= E[r_p] \\
    w' 1 &= 1 \\
    w' &\geq 0
\end{align*}
\]

Such short sale restricted minimum variance portfolio portfolios are much harder to deal with analytically, since they do not admit a general solution, one rather has to investigate the Kuhn-Tucker conditions for corner solutions etc. To deal with this problem in practice one will use a subroutine for solving constrained optimization problems.

6.10 The Sharpe Ratio

The Sharpe ratio of a given portfolio \( p \) is defined as

\[
S_p = \frac{E[r_p] - r_f}{\sigma(r_p)}
\]

The Sharpe ratio \( S_p \) of a portfolio \( p \) is the slope of the line in mean-standard deviations space from the risk free rate through \( p \). Note that in the case with a risk free asset, the tangency portfolio has the maximal Sharpe Ratio on the efficient frontier.
6.11 Equilibrium: CAPM

Under certain additional assumptions, an economy of mean variance optimizers will aggregate to an economy where the Capital Asset Pricing Model (CAPM) holds. Under the CAPM, any asset returns will satisfy:

\[ E[r_i] = r_f + \beta_i (E[r_m] - r_f) \]

where \( r_i \) is the return on asset \( i \), \( r_f \) the return on the risk free asset,

6.11.1 Treynor

\[ T_p = \frac{r_p - r_f}{\beta_p} \]

6.11.2 Jensen

\[ \alpha_p = r_p - (r_f + \beta_p (r_m - r_f)) \]

6.12 Working with Mean Variance and CAPM

The computational problems in mean variance optimization and the CAPM are not major, except for the case of short sales constrained portfolios (quadratic programming). The issues of more concern is estimation of parameters such as covariances and betas.

function s = sharpe(r, rf)
    s = (mean(r) - mean(rf))/std(r-rf);
endfunction

Matlab Code 6.2: Sharpe Ratio

function t = treynor(r, rm, rf)
    beta = cov(r,rm)/var(rm);
    t = (mean(r-rf))/beta;
endfunction

Matlab Code 6.3: Treynor Ratio

function alpha = jensen(r, rm, rf)
    beta = cov(r,rm)/var(rm);
    alpha = mean(r) - (rf+beta*(mean(rm)-rf));
endfunction

Matlab Code 6.4: Jensens alpha

Readings and Sources  The classical sources for this material are Merton (1972) and Roll (1977a). (Huang and Litzenberger, 1988, Ch 3) has a good textbook discussion of it.
6.13 Mean variance analysis using matrix libraries

Let us now consider how to implement mean variance analysis in C++. As shown using Matlab, the calculations are relatively simple matrix expressions. To implement the calculations in C++ the best way of doing it is to use a linear algebra class to deal with the calculations. For illustrative purposes we will show usage of two different matrix classes, Newmat and IT++. These classes are described in more detail in an appendix, with some references to how to obtain and install them.

In C++ Code 6.1 and C++ Code 6.2 we show how to do the basic mean variance calculations, calculating means and standard deviations, using the two classes. Note the similarity to using Matlab in the way the matrix and vector multiplications are carried out. Note also an important difference between the two classes. In IT++ the indexing of elements start at zero, the usual C++ way. Hence, when addressing element tmp(0,0) in the IT++ example we are pulling the first element. In Newmat the default indexing starts at one, the Matlab way. Therefore, in addressing tmp(1,1) in the Newmat example we are also pulling the first element. This serves as a warning to read the documentation carefully, the “off by one” error is a very common occurrence when mixing libraries like this.

```cpp
#include <cmath>
using namespace std;

#include <itpp/itbase.h>
using namespace itpp;

double mv_calculate_mean(const vec& e, const vec& w) {
    vec tmp = e.transpose() * w;
    return tmp(0);
};

double mv_calculate_variance(const mat& V, const vec& w) {
    mat tmp = w.transpose() * V * w;
    return tmp(0,0);
};

double mv_calculate_st_dev(const mat& V, const vec& w) {
    double var = mv_calculate_variance(V, w);
    return sqrt(var);
};
```

C++ Code 6.1: Mean variance calculations using IT++

Example

Mean variance calculations.

\[
\mathbf{e} = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}
\]

\[
\mathbf{V} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}
\]

Calculate mean, variance and stdev for portfolio

\[
\mathbf{w} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}
\]
#include <cmath>
using namespace std;

#include "newmat.h"
using namespace NEWMAT;

double mv_calculate_mean(const Matrix& e, const Matrix& w) {
    Matrix tmp = e.t() * w;
    return tmp(1,1);
};

double mv_calculate_variance(const Matrix& V, const Matrix& w) {
    Matrix tmp = w.t() * V * w;
    return tmp(1,1);
};

double mv_calculate_st_dev(const Matrix& V, const Matrix& w) {
    double var = mv_calculate_variance(V,w);
    return sqrt(var);
};

C++ Code 6.2: Mean variance calculations using Newmat

C++ program:

    cout << "Simple example of mean variance calculations " << endl;
    Matrix e(2,1);
    e(1,1)=0.05; e(2,1)=0.1;
    Matrix V(2,2);
    V(1,1)=1.0; V(2,1)=0.0;
    V(1,2)=0.0; V(2,2)=1.0;
    Matrix w(2,1);
    w(1,1)=0.5;
    w(2,1)=0.5;
    cout << " mean " << mv_calculate_mean(e,w) << endl;
    cout << " variance " << mv_calculate_variance(V,w) << endl;
    cout << " stdev " << mv_calculate_st_dev(V,w) << endl;

Output from C++ program:

    Simple example of mean variance calculations
    mean 0.075
    variance 0.5
    stdev 0.707107
In C++ Code 6.4 and C++ Code 6.3 we show how to calculate the mean variance optimal portfolio for a given required return. This is the case where there are no constraints on the weight, and we use the analytical solution directly.

```cpp
#include "newmat.h"
using namespace NEWMAT;

ReturnMatrix mv_calculate_portfolio_given_mean_unconstrained(const Matrix& e,
                                                          const Matrix& V,
                                                          const double& r){
    int no_assets=e.Nrows();
    Matrix ones = Matrix(no_assets,1), for (int i=0;i<no_assets;++i){ones.element(i,0) = 1;};
    Matrix Vinv = V.i(); // inverse of V
    Matrix A = (ones.t()*Vinv*e); double a = A.element(0,0);
    Matrix B = e.t()*Vinv*e; double b = B.element(0,0);
    Matrix C = ones.t()*Vinv*ones; double c = C.element(0,0);
    double d = b*c - a*a;
    Matrix Vinv1=Vinv*ones;
    Matrix Vinve=Vinv*e;
    Matrix g = (Vinv1*b - Vinve*a)*(1.0/d);
    Matrix h = (Vinve*c - Vinv1*a)*(1.0/d);
    Matrix w = g + h*r;
    w.Release();
    return w;
};
```

**C++ Code 6.3:** Calculating the unconstrained frontier portfolio given an expected return using Newmat

**Example**

Mean variance calculations.

\[
  \mathbf{e} = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}
\]

\[
  \mathbf{V} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}
\]

Find the optimal minimum variance portfolio with return \( r = 0.075 \).

**C++ program:**

```cpp
cout << "Testing portfolio calculation " << endl;
Matrix e(2,1);
e(1,1)=0.05; e(2,1)=0.1;
Matrix V(2,2);
V(1,1)=1.0; V(2,1)=0.0;
V(1,2)=0.0; V(2,2)=1.0;
double r=0.075;
Matrix w = mv_calculate_portfolio_given_mean_unconstrained(e,V,r);
cout << " suggested portfolio: " << endl;
cout << " w1 = " << w(1,1) << " w2 = " << w(2,1) << endl;
```

**Output from C++ program:**

Testing portfolio calculation

suggested portfolio:  w1 = 0.5  w2 = 0.5
```cpp
#include <itpp/itbase.h>
using namespace itpp;

mat mv_calculate_portfolio_given_mean_unconstrained(const vec & e,
                                                      const mat & V,
                                                      const double & r)
{
    int no_assets = e.size();
    vec one = ones(no_assets);
    mat Vinv = inv(V); // inverse of V
    mat A = one.transpose()*Vinv*e;
    double a = A(0,0);
    mat B = e.transpose()*Vinv*e;
    double b = B(0,0);
    mat C = one.transpose()*Vinv*one;
    double c = C(0,0);
    double d = b*c - a*a;
    mat Vinv1=Vinv*one;
    mat Vinv=Vinv*e;
    mat g = (Vinv1*b - Vinv*e*a)*(1.0/d);
    mat h = (Vinv*e - Vinv1*e*a)*(1.0/d);
    mat w = g + h*r;
    return w;
}
```

**C++ Code 6.4:** Calculating the unconstrained frontier portfolio given an expected return using IT++
Chapter 7

Futures algorithms.

Contents

7.1 Pricing of futures contract. ........................................... 81

In this we discuss algorithms used in valuing futures contracts.

7.1 Pricing of futures contract.

The futures price of an asset without payouts is the future value of the current price of the asset.

\[ f_t = e^{r(T-t)} S_t \]

```cpp
#include <cmath>
using namespace std;

double futures_price(const double& S, // current price of underlying asset
                     const double& r, // risk free interest rate
                     const double& time_to_maturity) {
    return exp(r*time_to_maturity)*S;
}
```

C++ Code 7.1: Futures price

Example

Let \( S = 100 \) and \( r = 10\% \). What is the futures price for a contract with time to maturity of half a year?

```cpp
double S=100; double r=0.10; double time=0.5;
cout << "futures price = " << futures_price(S,r,time) << endl;
```

Output from C++ program:

futures price = 105.127
Chapter 8

Binomial option pricing

Contents

8.1 Options . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 82
8.2 Pricing . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 82
8.3 Multiperiod binomial pricing . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 85

8.1 Options

Option and other derivative pricing is one of the prime “success stories” of modern finance. An option is a derivative security, the cash flows from the security is a function of the price of some other security, typically called the underlying security. A call option is a right, but not obligation, to buy a given quantity of the underlying security at a given price, called the exercise price \( K \), within a certain time interval. A put option is the right, but not obligation, to sell a given quantity of the underlying security to an agreed exercise price within a given time interval. If an option can only be exercised (used) at a given date (the time interval is one day), the option is called an European Option. If the option can be used in a whole time period up to a given date, the option is called American.

An option will only be used if it is valuable to the option holder. In the case of a call option, this is when the exercise price \( K \) is lower than the price one alternatively could buy the underlying security for, which is the current price of the underlying security. Hence, options have never negative cash flows at maturity. Thus, for anybody to be willing to offer an option, they must have a cost when entered into. This cost, or price, is typically called an option premium. As notation, let \( C \) signify the price of a call option, \( P \) the price of a put option and \( S \) the price of the underlying security. All of these prices are indexed by time. We typically let \( 0 \) be “now” and \( T \) the final maturity date of the option. From the definition of the options, it is clear that at their last possible exercise date, the maturity date, they have cash flows.

\[
C_T = \max(0, S_T - K)
\]

\[
P_T = \max(0, K - S_T)
\]

The challenge of option pricing is to determine the option premia \( C_0 \) (call price) and \( P_0 \) (put price).

8.2 Pricing

All pricing uses that the cashflows from the derivative is a direct function of the price of the underlying security. Pricing can therefore be done relative to the price of the underlying security. To price options
it is necessary to make assumptions about the probability distribution of movements of the underlying security. We start by considering this in a particularly simple framework, the binomial assumption. The price of the underlying is currently $S_0$. The price can next period only take on two values, $S_u$ and $S_d$.

If one can find all possible future “states,” an enumeration of all possibilities, one can value a security by constructing artificial “probabilities”, called “state price probabilities,” which one use to find an artificial expected value of the underlying security, which is then discounted at the risk free interest rate. The binomial framework is particularly simple, since there are only two possible states. If we find the “probability” $q$ of one state, we also find the probability of the other as $(1 - q)$. Equation 8.1 demonstrates this calculation for the underlying security.

$$S_0 = e^{-r} (q S_u + (1 - q) S_d)$$

(8.1)

Now, any derivative security based on this underlying security can be priced using the same “probability” $q$. The contribution of binomial option pricing is in actually calculating the number $q$. To do valuation, start by introducing constants $u$ and $d$ implicitly defined by $S_u = u S_0$ and $S_d = d S_0$, and you get a price process as illustrated in figure 8.1.

![Figure 8.1: Binomial Tree](image)

and calculate the artificial “probability” $q$ as

$$q = \frac{e^r - d}{u - d}$$

The price of a one-period call option in a binomial framework is shown in Formula 8.1 and implemented in C++ Code 8.1.

The “state price probability” $q$ is found by an assumption of no arbitrage opportunities. If one has the possibility of trading in the underlying security and a risk free bond, it is possible to create a portfolio of these two assets that exactly duplicates the future payoffs of the derivative security. Since this portfolio has the same future payoff as the derivative, the price of the derivative has to equal the cost of the duplicating portfolio. Working out the algebra of this, one can find the expression for $q$ as the function of the up and down movements $u$ and $d$.

Exercise 8.1.

The price of the underlying security follows the binomial process.
\[ C_u = \max(0, S_u - K) \]
\[ C_d = \max(0, S_d - K) \]
\[ C_0 = e^{-r} (qC_u + (1 - q)C_d) \]
\[ q = \frac{e^r - d}{u - d} \]

\( S_u = uS_0 \) and \( S_d = dS_0 \) are the possible values for the underlying security next period, \( u \) and \( d \) are constants, \( r \) is the (continuously compounded) risk free interest rate and \( K \) is the call option exercise price.

**Formula 8.1:** The single period binomial call option price

```cpp
#include <cmath>  // standard mathematical library
#include <algorithm>  // defining the max() operator
using namespace std;

double option_price_call_european_binomial_single_period(const double &S, // spot price
                                                         const double &X, // exercise price
                                                         const double &r, // interest rate (per period)
                                                         const double &u, // up movement
                                                         const double &d){  // down movement

double p_up = (exp(r)-d)/(u-d);
double p_down = 1.0-p_up;
double c_u = max(0.0,(u*S-X));
double c_d = max(0.0,(d*S-X));
double call_price = exp(-r)*(p_up*c_u+p_down*c_d);
return call_price;
};
```

**C++ Code 8.1:** Binomial European, one period

\( S_0 \)

\( S_u \)

\( S_d \)

A one period call option has payoffs

\[ C_u = \max(0, S_u - K) \]
\[ C_d = \max(0, S_d - K) \]

1. Show how one can combine a position in the underlying security with a position in risk free bonds to create a portfolio which exactly duplicates the payoffs from the call.

2. Use this result to show the one period pricing formula for a call option shown in formula 8.1.
8.3 Multiperiod binomial pricing

Of course, an assumption of only two possible future states next period is somewhat unrealistic, but if we iterate this assumption, and assume that every date, there are only two possible outcomes next date, but then, for each of these two outcomes, there is two new outcomes, as illustrated in the next figure:

\[
\begin{align*}
S_t & \\
& \downarrow \quad \downarrow \\
u S_t & \quad d S_t
\end{align*}
\]

\[
\begin{align*}
\nu(uS_t) &= \nu uS_t \\
\nu(dS_t) &= \nu dS_t \\
d(uS_t) &= dS_t \\
d(dS_t) &= ddS_t
\end{align*}
\]

Iterating this idea a few times more, the number of different terminal states increases markedly, and we get closer to a realistic distribution of future prices of the underlying at the terminal date. Note that a crucial assumption to get a picture like this is that the factors \( u \) and \( d \) are the same on each date.

\[
\begin{align*}
\text{Pricing in a setting like this is done by working backwards, starting at the terminal date. Here we know all the possible values of the underlying security. For each of these, we calculate the payoffs from the derivative, and find what the set of possible derivative prices is one period before. Given these, we can find the option one period before this again, and so on. Working ones way down to the root of the tree, the option price is found as the derivative price in the first node.}
\end{align*}
\]

For example, suppose we have two periods, and price a two period call option with exercise price \( K \).
First step: Find terminal payoffs of derivative security:

\[ C_{uu} = \max(0, S_{uu} - K) \]

\[ C_{du} = \max(0, duS - K) \]

\[ C_{dd} = \max(0, ddS - K) \]

Next step: Find the two possible call prices at time 1:

\[ C_u = e^{-r}(q C_{uu} + (1 - q)C_{ud}) \]

\[ C_d = e^{-r}(q C_{ud} + (1 - q)C_{dd}) \]

Final step: Using the two possible payoffs at time 1, \( C_u \) and \( C_d \), find option value at time 0:

\[ C_0 = e^{-r}(qC_u + (1 - q)C_d) \]

Thus, binomial pricing really concerns "rolling backward" in a binomial tree, and programming therefore concerns an efficient way of traversing such a tree. The obvious data structure for describing such a tree is shown in C++ Code 8.2, where the value in each node is calculated from finding out the number of up and down steps are used to get to the particular node.

Exercise 8.2.
#include <vector>
#include <cmath>
using namespace std;

vector< vector<double> > binomial_tree(const double& S0, const double& u, const double& d, const int& no_steps){
  vector< vector<double> > tree;
  for (int i=1;i<=no_steps;++i){
    vector<double> S(i);
    for (int j=0;j<i;++j){
      S[j] = S0*pow(u,j)*pow(d,i-j-1);
    }
    tree.push_back(S);
  }
  return tree;
}

C++ Code 8.2: Building a binomial tree

In terms of computational efficiency the approach of C++ Code 8.2 will not be optimal, since it requires a lot of calls to the pow() functional call. More efficient would be to carry out the tree building by doing the multiplication from the previous node, for example the j’th vector is the j-1’th vector times u, and then one need to add one more node by multiplying the lowest element by d.

1. Implement such an alternative tree building procedure.

Basing the recursive calculation of a derivative price on a triangular array structure as shown in C++ Code 8.2 is the most natural approach, but with some cleverness based on understanding the structure of the binomial tree, we can get away with the more efficient algorithm that is shown in C++ Code 8.3. Note that here we only use one vector<double>, not a triangular array as built above.

Example
Let $S = 100.0$, $K = 100.0$, $r = 0.025$, $u = 1.05$ and $d = 1/u$.

1. Price one and two period European Call options.

C++ program:
```cpp
double S = 100.0; double K = 100.0; double r = 0.025;
double u = 1.05; double d = 1/u;
cout << " one period european call = "
  << option_price_call_european_binomial_single_period(S,K,r,u,d) << endl;
int no_periods = 2;
cout << " two period european call = "
  << option_price_call_european_binomial_multi_period_given_ud(S,K,r,u,d,no_periods) << endl;
```

Output from C++ program:
```
one period european call = 3.64342
two period european call = 5.44255
```

Exercise 8.3.
Implement pricing of single and multi period binomial put options.

Further reading  The derivation of the single period binomial is shown in for example Bossaerts and Ødegaard (2001), Hull (2011) or McDonald (2013).
#include <cmath>  // standard mathematical library
#include <algorithm>  // defining the max() operator
#include <vector>  // STL vector templates
using namespace std;

double option_price_call_european_binomial_multi_period_given_ud(const double& S, const double& K, const double& r, const double& u, const double& d, const int& no_periods) {
    double Rinv = exp(-r);  // inverse of interest rate
    double uu = u * u;
    double p_up = (exp(r) - d) / (u - d);
    double p_down = 1.0 - p_up;
    vector<double> prices(no_periods + 1);  // price of underlying
    prices[0] = pow(d, no_periods);  // fill in the endnodes.
    for (int i = 1; i <= no_periods; ++i) { prices[i] = uu * prices[i - 1]; }
    vector<double> call_values(no_periods + 1);  // value of corresponding call
    for (int i = 0; i <= no_periods; ++i) { call_values[i] = max(0.0, (prices[i] - K)); }  // call payoffs at maturity
    for (int step = no_periods - 1; step >= 0; --step) {
        for (int i = 0; i <= step; ++i) {
            call_values[i] = (p_up * call_values[i + 1] + p_down * call_values[i]) * Rinv;
        }
    }
    return call_values[0];
}

C++ Code 8.3: Binomial multi-period pricing of European call option
Chapter 9

Basic Option Pricing, the Black Scholes formula

Contents

9.1 The formula ................................................................. 90
9.2 Understanding the why’s of the formula ............................... 92
  9.2.1 The original Black Scholes analysis ............................ 93
  9.2.2 The limit of a binomial case ...................................... 93
  9.2.3 The representative agent framework ............................ 93
9.3 Partial derivatives. ......................................................... 93
  9.3.1 Delta ................................................................. 93
  9.3.2 Other Derivatives ................................................ 94
  9.3.3 Implied Volatility. ................................................ 96
9.4 References ............................................................... 98

The pricing of options and related instruments has been a major breakthrough for the use of financial theory in practical application. Since the original papers of Black and Scholes (1973) and Merton (1973), there has been a wealth of practical and theoretical applications. We will now consider the original Black Scholes formula for pricing options, how it is calculated and used. For the basic intuition about option pricing the reader should first read the discussion of the binomial model in the previous chapter, as that is a much better environment for understanding what is actually calculated.

An option is a derivative security, its value depends on the value, or price, of some other underlying security, called the underlying security. Let $S$ denote the value, or price, of this underlying security. We need to keep track of what time this price is observed at, so let $S_t$ denote that the price is observed at time $t$. A call (put) option gives the holder the right, but not the obligation, to buy (sell) some underlying asset at a given price $K$, called the exercise price, on or before some given date $T$. If the option is a so called European option, it can only be used (exercised) at the maturity date. If the option is of the so called American type, it can be used (exercised) at any date up to and including the maturity date $T$. If exercised at time $T$, a call option provides payoff

$$C_T = \max(0, S_T - K)$$

and a put option provides payoff

$$P_T = \max(0, K - S_T)$$

The Black Scholes formulas provides analytical solutions for European put and call options, options which can only be exercised at the options maturity date. Black and Scholes showed that the additional
information needed to price the option is the (continuously compounded) risk free interest rate \( r \), the variability of the underlying asset, measured by the standard deviation \( \sigma \) of (log) price changes, and the time to maturity \((T - t)\) of the option, measured in years. The original formula was derived under the assumption that there are no payouts, such as stock dividends, coming from the underlying security during the life of the option. Such payouts will affect option values, as will become apparent later.

### 9.1 The formula

Formula 9.1 gives the exact formula for a call option, and the calculation of the same call option is shown in C++ Code 9.1 and Matlab Code 9.1.

\[
c = SN(d_1) - Ke^{-r(T-t)}N(d_2)
\]

where

\[
d_1 = \frac{\ln \left( \frac{S}{K} \right) + \left( r + \frac{1}{2} \sigma^2 \right)(T - t)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t}
\]

and

\[
d_2 = d_1 - \sigma \sqrt{T-t}
\]

Alternatively one can calculate \( d_1 \) and \( d_2 \) as

\[
d_1 = \frac{\ln \left( \frac{S}{K} \right) + r(T - t)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t}
\]

\[
d_2 = \frac{\ln \left( \frac{S}{K} \right) + r(T - t)}{\sigma \sqrt{T-t}} - \frac{1}{2} \sigma \sqrt{T-t}
\]

\( S \) is the price of the underlying security, \( K \) the exercise price, \( r \) the (continuously compounded) risk free interest rate, \( \sigma \) the standard deviation of the underlying asset, \( t \) the current date, \( T \) the maturity date, \( T - t \) the time to maturity for the option and \( N(\cdot) \) the cumulative normal distribution.

**Formula 9.1:** The Black Scholes formula

```cpp
#include <cmath> // mathematical C library
#include "normdist.h" // the calculation of the cumulative normal distribution

double option_price_call_black_scholes(const double& S, // spot (underlying) price
const double& K, // strike (exercise) price,
const double& r, // interest rate
const double& sigma, // volatility
const double& time) { // time to maturity

double time_sqrt = sqrt(time);
double d1 = (log(S/K)+r*time)/(sigma*time_sqrt)+0.5*sigma*time_sqrt;
double d2 = d1-(sigma*time_sqrt);
return S*N(d1) - K*exp(-r*time)*N(d2);
};
```

**C++ Code 9.1:** Price of European call option using the Black Scholes formula
function c = black_scholes_call(S,K,r,sigma,time)
    time_sqrt = sqrt(time);
    d1 = (log(S/K)+r*time)/(sigma*time_sqrt)+0.5*sigma*time_sqrt;
    d2 = d1-(sigma*time_sqrt);
    c = S * normcdf(d1) - K * exp(-r*time) * normcdf(d2);
endfunction

Matlab Code 9.1: Price of European call option using the Black Scholes formula
Example

Stock in company XYZ is currently trading at 50. Consider a call option on XYZ stock with an exercise price of $K = 50$ and time to maturity of 6 months. The volatility of XYZ stock has been estimated to be $\sigma = 30\%$. The current risk free interest rate (with continuous compounding) for six month borrowing is 10%.

1. Determine the option price.

To calculate the price of this option we use the Black Scholes formula with inputs $S = 50$, $K = 50$, $r = 0.10$, $\sigma = 0.3$ and $(T - t) = 0.5$.

Matlab program:

```matlab
S = 100;
K = 100;
r = 0.1;
sigma = 0.1;
time = 1;
c = black_scholes_call(S,K,r,sigma,time)
```

Output from Matlab program:

```
c = 10.308
```

C++ program:

```cpp
double S = 50; double K = 50; double r = 0.10;
double sigma = 0.30; double time=0.50;
cout << " Black Scholes call price = ";
cout << option_price_call_black_scholes(S, K , r, sigma, time) << endl;
```

Output from C++ program:

```
Black Scholes call price = 5.45325
```

Exercise 9.1.

The Black Scholes price for a put option is:

\[
p = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)
\]

where $d_1$ and $d_2$ are as for the call option:

\[
d_1 = \frac{\ln \left( \frac{S}{X} \right) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}
\]

\[
d_2 = d_1 - \sigma\sqrt{T - t},
\]

$S$ is the price of the underlying security, $K$ the exercise price, $r$ the (continuously compounded) risk free interest rate, $\sigma$ the standard deviation of the underlying asset, $T - t$ the time to maturity for the option and $N(\cdot)$ the cumulative normal distribution.

1. Implement this formula.

9.2 Understanding the why’s of the formula

To get some understanding of the Black Scholes formula and why it works will need to delve in some detail into the mathematics underlying its derivation. It does not help that there are a number of ways to prove the Black Scholes formula, depending on the setup. As it turns out, two of these ways are important to understand for computational purposes, the original Black Scholes continuous time way, and the “limit of a binomial process” way of Cox, Ross, and Rubinstein (1979).
9.2.1 The original Black Scholes analysis

The primary assumption underlying the Black Scholes analysis concerns the stochastic process governing the price of the underlying asset. The price of the underlying asset, $S$, is assumed to follow a geometric Brownian Motion process, conveniently written in either of the shorthand forms

$$dS = \mu S dt + \sigma S dZ$$

or

$$\frac{dS}{S} = \mu dt + \sigma dZ$$

where $\mu$ and $\sigma$ are constants, and $Z$ is Brownian motion.

Using Ito's lemma, the assumption of no arbitrage, and the ability to trade continuously, Black and Scholes showed that the price of any contingent claim written on the underlying must solve the partial differential equation (9.1).

$$\frac{\partial f}{\partial S} rS + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 = rf$$

(9.1)

For any particular contingent claim, the terms of the claim will give a number of boundary conditions that determines the form of the pricing formula.

The pde given in equation (9.1), with the boundary condition $c_T = \max(0, S_T - K)$ was shown by Black and Scholes to have an analytical solution of functional form shown in the Black Scholes formula 9.1.

9.2.2 The limit of a binomial case

Another is to use the limit of a binomial process (Cox et al., 1979). The latter is particularly interesting, as it allows us to link the Black Scholes formula to the binomial, allowing the binomial framework to be used as an approximation.

9.2.3 The representative agent framework

A final way to show the BS formula to assume a representative agent and lognormality as was done in Rubinstein (1976).

9.3 Partial derivatives.

In trading of options, a number of partial derivatives of the option price formula is important.

9.3.1 Delta

The first derivative of the option price with respect to the price of the underlying security is called the delta of the option price. It is the derivative most people will run into, since it is important in hedging of options.

$$\frac{\partial c}{\partial S} = N(d_1)$$

Code 9.2 shows the calculation of the delta for a call option.
#include <cmath>
#include "normdist.h"

double option_price_delta_call_black_scholes(const double & S, // spot price
   const double & K, // Strike (exercise) price,
   const double & r, // interest rate
   const double & sigma, // volatility
   const double & time){ // time to maturity

double time_sqrt = sqrt(time);

double d1 = (log(S/K)+r*time)/(sigma*time_sqrt) + 0.5*sigma*time_sqrt;

double delta = N(d1);
return delta;
};

C++ Code 9.2: Calculating the delta of the Black Scholes call option price

9.3.2 Other Derivatives

The remaining derivatives are more seldom used, but all of them are relevant. All of them are listed in formula 9.3.2.

Delta (\Delta)
\[ \Delta = \frac{\partial c}{\partial S} = N(d_1) \]

Gamma (\Gamma)
\[ \frac{\partial^2 c}{\partial S^2} = \frac{n(d_1)}{S\sigma\sqrt{T-t}} \]

Theta (\Theta) (careful about which of these you want)
\[ \frac{\partial c}{\partial T} = Sn(d_1)\frac{1}{2}\sigma^2\frac{1}{\sqrt{T-t}} + rKe^{-r(T-t)}N(d_2) \]
\[ \frac{\partial c}{\partial t} = -Sn(d_1)\frac{1}{2}\sigma^2\frac{1}{\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2) \]

Vega
\[ \frac{\partial c}{\partial \sigma} = S\sqrt{T-t}n(d_1) \]

Rho (\rho)
\[ \frac{\partial c}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2) \]

Formula 9.2: Partial derivatives of the Black Scholes call option formula

The calculation of all of these partial derivatives for a call option is shown in C++ Code 9.3.

Example
Consider again six month call options on XYZ stock. The option matures 6 months from now, at which time the holder of the option can receive one unit of the underlying security by paying the exercise price of \( K = 50 \). The current price of the underlying security is \( S = 50 \). The volatility of the underlying security is given as \( \sigma = 30\% \). The current risk free interest rate (with continuous compounding) for six month borrowing is 10\%.
```cpp
#include <cmath>
#include "normdist.h"

using namespace std;

void option_pricepartials_call_black_scholes(
    const double &S, // spot price
    const double &K, // Strike (exercise) price,
    const double &r, // interest rate
    const double &sigma, // volatility
    const double &time, // time to maturity
    double &Delta, // partial wrt S
    double &Gamma, // second prt wrt S
    double &Theta, // partial wrt time
    double &Vega, // partial wrt sigma
    double &Rho) { // partial wrt r

    double time_sqrt = sqrt(time);
    double d1 = (log(S/K)+r*time)/(sigma*time_sqrt) + 0.5*sigma*time_sqrt;
    double d2 = d1 - (sigma*time_sqrt);
    Delta = N(d1);
    Gamma = n(d1)/(S*sigma*time_sqrt);
    Theta = (S*sigma*n(d1))/(2*time_sqrt) - r*K*exp(-r*time)*N(d2);
    Vega = S * time_sqrt * n(d1);
    Rho = K*time*exp(-r*time)*N(d2);
}
```

C++ Code 9.3: Calculating the partial derivatives of a Black Scholes call option

1. Calculate the partial derivatives (Delta, Gamma, Theta, Vega and Rho) for this option.

To calculate the partial derivatives we use inputs \( S = 50, K = 50, r = 0.10, \sigma = 0.3 \text{ and } (T - t) = 0.5 \).

```cpp
C++ program:

    cout << " Black Scholes call partial derivatives " << endl;
    double S = 50; double K = 50; double r = 0.10;
    double sigma = 0.30; double time=0.50;
    double Delta, Gamma, Theta, Vega, Rho;
    option_pricepartials_call_black_scholes(S,K,r,sigma, time, Delta, Gamma, Theta, Vega, Rho);
    cout << " Delta = " << Delta << endl;
    cout << " Gamma = " << Gamma << endl;
    cout << " Theta = " << Theta << endl;
    cout << " Vega = " << Vega << endl;
    cout << " Rho = " << Rho << endl;

Output from C++ program:

Black Scholes call partial derivatives
Delta = 0.633737
Gamma = 0.0354789
Theta = -6.61473
Vega = 13.3046
Rho = 13.1168
```
9.3.3 Implied Volatility.

In calculation of the option pricing formulas, in particular the Black Scholes formula, the only unknown is the standard deviation of the underlying stock. A common problem in option pricing is to find the implied volatility, given the observed price quoted in the market. For example, given \( c_0 \), the price of a call option, the following equation should be solved for the value of \( \sigma \):

\[
c_0 = c(S, K, r, \sigma, T - t)
\]

Unfortunately, this equation has no closed form solution, which means the equation must be numerically solved to find \( \sigma \). What is probably the algorithmic simplest way to solve this is to use a binomial search algorithm, which is implemented in C++ Code 9.4. We start by bracketing the sigma by finding a high sigma that makes the BS price higher than the observed price, and then, given the bracketing interval, we search for the volatility in a systematic way. shows such a calculation.

```cpp
#include <cmath>
#include "fin_recipes.h"

double option_price_implied_volatility_call_black_scholes_bisections(const double& S, const double& K, const double& r, const double& time, const double& option_price){
    if (option_price<0.99*(S-K*exp(-time*r))) { // check for arbitrage violations.
        return 0.0; // Option price is too low if this happens
    }
    // simple binomial search for the implied volatility.
    // relies on the value of the option increasing in volatility
    const double ACCURACY = 1.0e-5; // make this smaller for higher accuracy
    const int MAX_ITERATIONS = 100;
    const double HIGH_VALUE = 1e10;
    const double ERROR = -1e40;
    // want to bracket sigma. first find a maximum sigma by finding a sigma
    // with an estimated price higher than the actual price.
    double sigma_low=1e-5;
    double sigma_high=0.3;
    double price = option_price_call_black_scholes(S,K,r,sigma_high,time);
    while (price < option_price) {
        sigma_high = 2.0 * sigma_high; // keep doubling.
        price = option_price_call_black_scholes(S,K,r,sigma_high,time);
        if (sigma_high>HIGH_VALUE) return ERROR; // panic, something wrong.
    }
    for (int i=0;i<MAX_ITERATIONS;++i){
        double sigma = (sigma_low+sigma_high)*0.5;
        price = option_price_call_black_scholes(S,K,r,sigma,time);
        double test = (price−option_price);
        if (fabs(test)<ACCURACY) { return sigma; }
        if (test < 0.0) { sigma_low = sigma; }
        else { sigma_high = sigma; }
    }
    return ERROR;
}
```

C++ Code 9.4: Calculation of implied volatility of Black Scholes using bisections

Instead of this simple bracketing, which is actually pretty fast, and will (almost) always find the solution, we can use the Newton–Raphson formula for finding the root of an equation in a single variable. The general description of this method starts with a function \( f() \) for which we want to find a root.

\[
f(x) = 0.
\]
The function \( f() \) needs to be differentiable. Given a first guess \( x_0 \), iterate by

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}
\]

until

\[
|f(x_i)| < \epsilon
\]

where \( \epsilon \) is the desired accuracy.\(^1\)

In our case

\[
f(x) = c_{obs} - c_{BS}(\sigma)
\]

and, each new iteration will calculate

\[
\sigma_{i+1} = \sigma_i + \frac{c_{obs} - c_{BS}(\sigma_i)}{-\frac{\partial c_{BS}(\sigma_i)}{\partial \sigma}}
\]

C++ Code 9.5 shows the calculation of implied volatility using Newton-Raphson.

```cpp
#include "fin_recipes.h"
#include "normdist.h"
#include <cmath>
#include <iostream>

double option_price_implied_volatility_call_black_scholes_newton(const double& S,
const double& K,
const double& r,
const double& time,
const double& option_price) {
    if (option_price<0.99*(S-K*exp(-time*r))) { // check for arbitrage violations. Option price is too low if this happens
        return 0.0;
    }
    const int MAX_ITERATIONS = 100;
    const double ACCURACY = 1.0e-5;
    double t_sqrt = sqrt(time);
    double sigma = (option_price/S)/(0.398*t_sqrt); // find initial value
    for (int i=0;i<MAX_ITERATIONS;i++){
        double price = option_price_call_black_scholes(S,K,r,sigma,time);
        double diff = option_price - price;
        if (fabs(diff)<ACCURACY) return sigma;
        double d1 = (log(S/K)+r*time)/(sigma*t_sqrt) + 0.5*sigma*t_sqrt;
        double vega = S * t_sqrt * n(d1);
        sigma = sigma + diff/vega;
    }
    return -99e10; // something screwy happened, should throw exception
}
```

C++ Code 9.5: Calculation of implied volatility of Black Scholes using Newton-Raphson

Note that to use Newton-Raphson we need the derivative of the option price. For the Black-Scholes
formula this is known, and we can use this. But for pricing formulas like the binomial, where the partial
derivatives are not that easy to calculate, simple bisection is the preferred algorithm.

Example

Consider again six month call options on XYZ stock. The option matures 6 months from now, at which time
the holder of the option can receive one unit of the underlying security by paying the exercise price of \( K = 50 \).

\(^1\)For further discussion of the Newton-Raphson formula and bracketing, a good source is chapter 9 of Press et al. (1992)
The current price of the underlying security is \( S = 50 \). The current risk free interest rate (with continuous compounding) for six month borrowing is 10%.

1. The current option price is \( C = 2.5 \). Determine the volatility implicit in this price.

The implied volatility is the \( \sigma \) which, input in the Black Scholes formula with these other inputs, will produce an option price of \( C = 2.5 \).
To calculate we use inputs \( S = 50, K = 50, r = 0.10 \) and \( (T-t) = 0.5 \).

<table>
<thead>
<tr>
<th>C++ program:</th>
</tr>
</thead>
<tbody>
<tr>
<td>double S = 50; double K = 50; double r = 0.10; double time=0.50;</td>
</tr>
<tr>
<td>double C=2.5;</td>
</tr>
<tr>
<td>cout &lt;&lt; &quot; Black Scholes implied volatility using Newton search = &quot;;</td>
</tr>
<tr>
<td>cout &lt;&lt; option_price_implied_volatility_call_black_scholes_newton(S,K,r,time,C) &lt;&lt; endl;</td>
</tr>
<tr>
<td>cout &lt;&lt; &quot; Black Scholes implied volatility using bisections = &quot;;</td>
</tr>
<tr>
<td>cout &lt;&lt; option_price_implied_volatility_call_black_scholes_bisections(S,K,r,time,C) &lt;&lt; endl;</td>
</tr>
</tbody>
</table>

Output from C++ program:
Black Scholes implied volatility using Newton search = 0.0500427
Black Scholes implied volatility using bisections = 0.0500419

9.4 References
Black and Scholes (1973) Merton (1973)
Gray and Gray (2001) has a simple proof of the formula.
Chapter 10

Warrants

Contents

10.1 Warrant value in terms of assets ......................................................... 99
10.2 Valuing warrants when observing the stock value ................................. 100
10.3 Readings ......................................................................................... 101

A warrant is an option-like security on equity, but it is issued by the same company which has issued the equity, and when a warrant is exercised, a new stock is issued. This new stock is issued at the warrant strike price, which is lower than the current stock price (If it wasn’t the warrant would not be exercised.) Since the new stock is a fractional right to all cashflows, this stock issue waters out, or dilutes, the equity in a company. The degree of dilution is a function of how many warrants are issued.

10.1 Warrant value in terms of assets

Let $K$ be the strike price, $n$ the number of shares outstanding and $m$ the number of warrants issued. Assume each warrant is for 1 new share, and let $A_t$ be the current asset value of firm. Suppose all warrants are exercised simultaneously. Then the assets of the firm increase by the number of warrants times the strike price of the warrant.

$$A_t + mK,$$

but this new asset value is spread over more shares, since each exercised warrant is now an equity. The assets of the firm is spread over all shares, hence each new share is worth:

$$\frac{A_t + mK}{m+n},$$

making each exercised warrant worth:

$$\frac{A_t + mK}{m+n} - K = \frac{n}{m+n} \left( \frac{A_t}{n} - K \right)$$

If we knew the current value of assets in the company, we could value the warrant in two steps:

1. Value the option using the Black Scholes formula and $\frac{A_t}{n}$ as the current stock price.
2. Multiply the resulting call price with $\frac{n}{m+n}$.

If we let $W_t$ be the warrant value, the above arguments are summarized as:

$$W_t = \frac{n}{n+m} C_{BS} \left( \frac{A}{n}, K, \sigma, r, (T-t) \right),$$
where $C_{BS}(\cdot)$ is the Black Scholes formula.

### 10.2 Valuing warrants when observing the stock value

However, one does not necessarily observe the asset value of the firm. Typically one only observes the equity value of the firm. If we let $S_t$ be the current stock price, the asset value is really:

$$A_t = nS_t + mW_t$$

Using the stock price, one would value the warrant as

$$W_t = \frac{n}{n + m}C_{BS}\left(\frac{nS_t + mW_t}{n}, K, \sigma, r, (T - t)\right)$$

or

$$W_t = \frac{n}{n + m}C_{BS}\left(S_t + \frac{m}{n}W_t, K, \sigma, r, (T - t)\right)$$

Note that this gives the value of $W_t$ as a function of $W_t$. One need to solve this equation numerically to find $W_t$.

The numerical solution for $W_t$ is done using the Newton-Rhapson method. Let

$$g(W_t) = W_t - \frac{n}{n + m}C_{BS}\left(S_t + \frac{m}{n}W_t, K, \sigma, r, (T - t)\right)$$

Starting with an initial guess for the warrant value $W_t^0$, the Newton-Rhapson method is that one iterates as follows

$$W_t^i = W_t^{i-1} - \frac{g(W_t^{i-1})}{g'(W_t^{i-1})},$$

where $i$ signifies iteration $i$, until the criterion function $g(W_t^{i-1})$ is below some given accuracy $\epsilon$. In this case

$$g'(W_t) = 1 - \frac{m}{m + n}N(d_1)$$

where

$$d_1 = \frac{\ln \left(\frac{S_t + \frac{m}{n}W_t}{K}\right) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}}$$

An obvious starting value is to set calculate the Black Scholes value using the current stock price, and multiply it with $\frac{m}{m + n}$.

C++ Code 10.1 implements this calculation.

**Example**

A stock is currently priced at $S = 48$. Consider warrants on the same company with exercise price $K = 40$ and time to maturity of six months. The company has $n = 10000$ shares outstanding, and has issued $m = 1000$ warrants. The current (continously compounded) risk free interest rate is 8%. Determine the current warrant price.
#include "fin_recipes.h"
#include "normdist.h"
#include <cmath>

const double EPSILON = 0.00001;

double warrant_price_adjusted_black_scholes(const double & S,
                                         const double & K,
                                         const double & r,
                                         const double & sigma,
                                         const double & time,
                                         const double & m, // number of warrants outstanding
                                         const double & n){ // number of shares outstanding

double time_sqrt = sqrt(time);

double w = (n/(n+m))*option_price_call_black_scholes(S,K,r,sigma,time);

double g = w-(n/(n+m))*option_price_call_black_scholes(S+(m/n)*w,K,r,sigma,time);

while (fabs(g)>EPSILON) {
    double d1 = (log((S+(m/n))/K)+r*time)/(sigma*time_sqrt)+0.5*sigma*time_sqrt;
    double gprime = 1-(m/n)*N(d1);
    w=w-g/gprime;
    g = w-(n/(n+m))*option_price_call_black_scholes(S+(m/n)*w,K,r,sigma,time);
}
return w;
}

C++ Code 10.1: Adjusted Black Scholes value for a Warrant

C++ program:

double S = 48; double K = 40; double r = 0.08; double sigma = 0.30;
double time = 0.5; double m = 1000; double n = 10000;
double w = warrant_price_adjusted_black_scholes(S,K,r,sigma, time, m, n);

Output from C++ program:

warrant price = 10.142

Exercise 10.1.

The solution method assumes that all warrants are exercised simultaneously. Can you see where this assumption is used?

10.3 Readings

McDonald (2013) and Hull (2011) are general references. A problem with warrants is that exercise of all warrants simultaneously is not necessarily optimal.

Press et al. (1992) discusses the Newton-Rhapson method for root finding.
Chapter 11
Extending the Black Scholes formula

Contents

11.1 Adjusting for payouts of the underlying. ......................................................... 102
   11.1.1 Continuous Payouts from underlying. .................................................... 102
   11.1.2 Dividends. .............................................................................................. 103
11.2 American options ............................................................................................... 104
   11.2.1 Exact american call formula when stock is paying one dividend. ............... 105
11.3 Options on futures ............................................................................................. 108
   11.3.1 Black’s model ......................................................................................... 108
11.4 Foreign Currency Options .................................................................................. 109
11.5 Perpetual puts and calls .................................................................................... 110
11.6 Readings ............................................................................................................ 111

11.1 Adjusting for payouts of the underlying.

For options on other financial instruments than stocks, we have to allow for the fact that the underlying
may have payouts during the life of the option. For example, in working with commodity options, there
is often some storage costs if one wanted to hedge the option by buying the underlying.

11.1.1 Continuous Payouts from underlying.

The simplest case is when the payouts are done continuously. To value an European option, a simple
adjustment to the Black Scholes formula is all that is needed. Let $q$ be the continuous payout of the
underlying commodity.

Call and put prices for European options are then given by formula 11.1, which are implemented in
C++ Code 11.1.

Exercise 11.1.

The price of a put on an underlying security with a continuous payout of $q$ is:

$$ p = Ke^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1) $$

1. Implement this formula.
\[ c = S e^{-q(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2) \]

where
\[ d_1 = \frac{\ln \left( \frac{S}{K} \right) + (r - q + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} \]
\[ d_2 = d_1 - \sigma \sqrt{T-t} \]

\( S \) is the price of the underlying security, \( K \) the exercise price, \( r \) the risk free interest rate, \( q \) the (continuous) payout and \( \sigma \) the standard deviation of the underlying asset, \( t \) the current date, \( T \) the maturity date, \( T-t \) the time to maturity for the option and \( N(\cdot) \) the cumulative normal distribution.

**Formula 11.1:** Analytical prices for European call option on underlying security having a payout of \( q \)

```
#include <cmath> // mathematical library
#include "normdist.h" // this defines the normal distribution
using namespace std;

double option_price_european_call_payout(const double& S, // spot price
                                        const double& X, // Strike (exercise) price,
                                        const double& r, // interest rate
                                        const double& q, // yield on underlying
                                        const double& sigma, // volatility
                                        const double& time) { // time to maturity

    double sigma_sqr = pow(sigma,2);
    double time_sqrt = sqrt(time);
    double d1 = (log(S/X) + (r-q + 0.5*sigma_sqr)*time)/(sigma*time_sqrt);
    double d2 = d1 - sigma*time_sqrt;
    double call_price = S * exp(-q*time)* N(d1) - X * exp(-r*time) * N(d2);
    return call_price;
}
```

C++ Code 11.1: Option price, continuous payout from underlying

### 11.1.2 Dividends.

A special case of payouts from the underlying security is stock options when the stock pays dividends. When the stock pays dividends, the pricing formula is adjusted, because the dividend changes the value of the underlying.

The case of continuous dividends is easiest to deal with. It corresponds to the continuous payouts we have looked at previously. The problem is the fact that most dividends are paid at discrete dates.

**European Options on dividend-paying stock.**

To adjust the price of an European option for known dividends, we merely subtract the present value of the dividends from the current price of the underlying asset in calculating the Black Scholes value.

**Example**

Consider a stock option with the following data given: \( S = 100 \), \( K = 100 \), \( r = 0.1 \), \( \sigma = 0.25 \), time to maturity is one year. dividend yield = 5%. Dividend payments at times 0.25 and 0.75.

- Determine the option price
American options are much harder to deal with than European ones. The problem is that it may be optimal to use (exercise) the option before the final expiry date. This optimal exercise policy will affect the value of the option, and the exercise policy needs to be known when solving the pde. However, the exercise policy is not known. There is therefore no general analytical solutions for American call and put options. There are some special cases. For American call options on assets that do not have any payouts, the American call price is the same as the European one, since the optimal exercise policy is to not exercise. For American puts this is not the case, it may pay to exercise them early. When the underlying asset has payouts, it may also pay to exercise an American call option early. There is one known known analytical price for American call options, which is the case of a call on a stock that pays a known dividend once during the life of the option, which is discussed next. In all other cases the American price has to be approximated using one of the techniques discussed in later chapters: Binomial approximation, numerical solution of the partial differential equation, or another numerical approximation.
11.2.1 Exact american call formula when stock is paying one dividend.

When a stock pays dividend, a call option on the stock may be optimally exercised just before the stock goes ex-dividend. While the general dividend problem is usually approximated somehow, for the special case of one dividend payment during the life of an option an analytical solution is available, due to Roll–Geske–Whaley.

If we let $S$ be the stock price, $K$ the exercise price, $D_1$ the amount of dividend paid, $t_1$ the time of dividend payment, $T$ the maturity date of option, we denote the time to dividend payment $\tau_1 = T - t_1$ and the time to maturity $\tau = T - t$.

A first check of early exercise is:

$$D_1 \leq K \left( 1 - e^{-r(T-t_1)} \right)$$

If this inequality is fulfilled, early exercise is not optimal, and the value of the option is

$$c(S - e^{-r(t_1-t)}D_1, K, r, \sigma, (T-t))$$

where $c(\cdot)$ is the regular Black Scholes formula.

If the inequality is not fulfilled, one performs the calculation shown in Formula 11.2 and implemented in \texttt{C++} Code 11.3.

$$C = (S - D_1 e^{-r(t_1-t)}) (N(b_1) + N(a_1, -b_1, \rho)) + K e^{-r(T-t)} N(a_2, -b_2, \rho) - (K - D_1) e^{-r(t_1-t)} N(b_2)$$

where

$$\rho = -\sqrt{\frac{(t_1-t)}{T-t}}$$

$$a_1 = \frac{\ln \left( \frac{S-D_1 e^{-r(t_1-t)}}{K} \right) + \left( r + \frac{1}{2} \sigma^2 \right)(t_1-t)}{\sigma \sqrt{T-t}}$$

$$a_2 = a_1 - \sigma \sqrt{T-t}$$

$$b_1 = \frac{\ln \left( \frac{S-D_1 e^{-r(t_1-t)}}{S} \right) + \left( r + \frac{1}{2} \sigma^2 \right)(t_1-t)}{\sigma \sqrt{(t_1-t)}}$$

$$b_2 = b_1 - \sigma \sqrt{T-t}$$

and $\tilde{S}$ solves

$$c(\tilde{S}, t_1) = \tilde{S} + D_1 - K$$

$S$ is the price of the underlying secrutity, $K$ the exercise price, $r$ the risk free interest rate, $D_1$ is the dividend amount and $\sigma$ the standard deviation of the underlying asset, $t$ the current date, $T$ the maturity date, $T-t$ the time to maturity for the option and $N(\cdot)$ the cumulative normal distribution. $N(\cdot)$ with one argument is the univariate normal cumulative distribution. $N(\cdot)$ with three arguments is the bivariate normal distribution with the correlation between the two normals given as the third argument.

**Formula 11.2:** Roll–Geske–Whaley price of american call option paying one fixed dividend

**Example**

Consider an option on a stock paying one dividend. The relevant data is $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$, $\tau = 1.0$, $\tau_1 = 0.5$ and $D_1 = 10$.

Price the option.
Exercise 11.2.

The Black approximation to the price of a call option paying a fixed dividend is an approximation to the value of the call. Suppose the dividend is paid as some date $t_1$ before the maturity date of the option $T$. Black's approximation calculates the value of two European options using the Black Scholes formula. One with expiry date equal to the ex-dividend date of the options. Another with expiry date equal to the option expiry, but the current price of the underlying security is adjusted down by the amount of the dividend.

1. Implement Black's approximation.
#include <cmath>
#include "normdist.h" // define the normal distribution functions
#include "fin_recipes.h" // the regular black sholes formula

double option_price_american_call_one_dividend(const double & S,  
    const double & K,  
    const double & r,  
    const double & sigma,  
    const double & tau,  
    const double & D1,  
    const double & tau1) {  
    if (D1 <= K * (1.0 - exp(-r*(tau - tau1)))) { // check for no exercise  
        return option_price_call_black_scholes(S - exp(-r*tau1)*D1,K,r,sigma,tau);  
    }  
    const double ACCURACY = 1e-6;  
    // decrease this for more accuracy  
    double sigma_sqr = sigma*sigma;  
    double tau_sqrt = sqrt(tau);  
    double tau1_sqrt = sqrt(tau1);  
    double rho = - sqrt(tau1/tau);  
    double S_bar = 0;  // first find the S_bar that solves c=S_bar+D1-K  
    double S_low = 0;  // the simplest: binomial search  
    double S_high = S;  // start by finding a very high S above S_bar  
    double c = option_price_call_black_scholes(S_high,K,r,sigma,tau-tau1);  
    double test = c-S_high-D1+K;  
    while ( (test>0.0) && (S_high<=1e10) ) {  
        S_high *= 2.0;  
        c = option_price_call_black_scholes(S_high,K,r,sigma,tau-tau1);  
        test = c-S_high-D1+K;  
    }  
    if (S_high>1e10) { // early exercise never optimal, find BS value  
        return option_price_call_black_scholes(S-D1*exp(-r*tau1),K,r,sigma,tau);  
    }  
    S_bar = 0.5 * S_high;  // now find S_bar that solves c=S_bar-D1+K  
    c = option_price_call_black_scholes(S_bar,K,r,sigma,tau-tau1);  
    test = c-S_bar-D1+K;  
    while ( (fabs(test)>ACCURACY) && ((S_high-S_low)>ACCURACY) ) {  
        if (test<0.0) { S_high = S_bar; }  
        else { S_low = S_bar; }  
        S_bar = 0.5 * (S_high + S_low);  
        c = option_price_call_black_scholes(S_bar,K,r,sigma,tau-tau1);  
        test = c-S_bar-D1+K;  
    }  
    double a1 = (log((S-D1*exp(-r*tau1))/K) + (r+0.5*sigma_sqr)*tau) / (sigma*tau_sqrt);  
    double a2 = a1 - sigma*tau_sqrt;  
    double b1 = (log((S-D1*exp(-r*tau1))/S_bar)+(r+0.5*sigma_sqr)*tau1) / (sigma*tau1_sqrt);  
    double b2 = b1 - sigma*tau1_sqrt;  
    double C = (S-D1*exp(-r*tau1)) * N(b1) + (S-D1*exp(-r*tau1)) * N(a1-b1,rho)  
        - (K*exp(-r*tau1)) * N(a2-b2,rho) - (K-D1)*exp(-r*tau1) * N(b2);  
    return C;  
}  

C++ Code 11.3: Option price, Roll–Geske–Whaley call formula for dividend paying stock
11.3 Options on futures

11.3.1 Black’s model

For an European option written on a futures contract, we use an adjustment of the Black Scholes solution, which was developed in Black (1976). Essentially we replace \( S_0 \) with \( e^{-r(T-t)}F \) in the Black Scholes formula, and get the formula shown in 11.3 and implemented in C++ Code 11.4.

\[
c = e^{-r(T-t)} (FN(d_1) - KN(d_2))
\]

where

\[
d_1 = \frac{\ln \left( \frac{F}{K} \right) + \frac{1}{2} \sigma^2 (T-t)}{\sigma \sqrt{T-t}}
\]

\[
d_2 = d_1 - \sigma \sqrt{T-t}
\]

\( F \) is the futures price, \( K \) is the exercise price, \( r \) the risk free interest rate, \( \sigma \) the volatility of the futures price, and \( T-t \) is the time to maturity of the option (in years).

**Formula 11.3:** Black's formula for the price of an European Call option with a futures contract as the underlying security

```cpp
#include <cmath>   // mathematics library
#include "normdist.h"   // normal distribution
using namespace std;

double futures_option_price_call_european_black(const double& F, // futures price
                                                  const double& K,  // exercise price
                                                  const double& r,  // interest rate
                                                  const double& sigma, // volatility
                                                  const double& time){ // time to maturity

double sigma_sqrt = sigma*sigma;
    double time_sqrt = sqrt(time);
    double d1 = (log(F/K) + 0.5 * sigma_sqrt * time) / (sigma * time_sqrt);
    double d2 = d1 - sigma * time_sqrt;
    return exp(-r*time)*(F * N(d1) - K * N(d2));
};
```

**C++ Code 11.4:** Price of European Call option on Futures contract

**Example**

Price a futures option in the Black setting. Information: \( F = 50 \), \( K = 45 \), \( r = 8\% \), \( \sigma = 0.2 \), and time to maturity is a half year.

**C++ program:**

```cpp
double F = 50.0; double K = 45.0;
    double r = 0.08; double sigma = 0.2;
    double time=0.5;
    cout << " european futures call option = ";
    futures_option_price_put_european_black(F,K,r,sigma,time) << endl;
```

**Output from C++ program:**

```
european futures call option = 0.851476
```

**Exercise 11.3.**
The Black formula for a put option on a futures contract is
\[ p = e^{-r(T-t)} (KN(-d_2) - FN(-d_1)) \]
where the variables are as defined for the call option.

1. Implement the put option price.

11.4 Foreign Currency Options

Another relatively simple adjustment of the Black Scholes formula occurs when the underlying security is a currency exchange rate (spot rate). In this case one adjusts the Black-Scholes equation for the interest-rate differential.

Let \( S \) be the spot exchange rate, and now let \( r \) be the domestic interest rate and \( r_f \) the foreign interest rate. \( \sigma \) is then the volatility of changes in the exchange rate. The calculation of the price of an European call option is then shown in formula 11.4 and implemented in C++. Code 11.5.

\[
c = Se^{-r_f(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2)
\]

where
\[
d_1 = \frac{\ln \left( \frac{S}{K} \right) + (r - r_f + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}
\]
\[
d_2 = d_1 - \sigma\sqrt{T-t}
\]

\( S \) is the spot exchange rate and \( K \) the exercise price. \( r \) is the domestic interest rate and \( r_f \) the foreign interest rate. \( \sigma \) is the volatility of changes in the exchange rate. \( T - t \) is the time to maturity for the option.

**Formula 11.4: European currency call**

```
#include <cmath>
#include "normdist.h" // define the normal distribution function

double currency_option_price_call_european( const double& S, // exchange_rate,
                                           const double& X, // exercise,
                                           const double& r, // r_domestic,
                                           const double& r_f, // r_foreign,
                                           const double& sigma, // volatility,
                                           const double& time){ // time to maturity

double sigma_sqr = sigma*sigma;
double time_sqrt = sqrt(time);
double d1 = (log(S/X) + (r-r_f+ (0.5*sigma_sqr))*(T-t))/(sigma*time_sqrt);
double d2 = d1 - sigma * time_sqrt;
return S * exp(-r_f*time) * N(d1) - X * exp(-r*time) * N(d2);
};
```

**C++ Code 11.5: European Futures Call option on currency**

**Example**

Price a European currency call given the following information \( S = 50, \ K = 52, \ r = 8\%, \ r_f = 5\%, \ \sigma = 20\% \) and time to maturity = 0.5 years.
**C++ program:**
```
#include <cmath>
using namespace std;

double option_price_american_perpetual_call(const double& S,
                                          const double& K,
                                          const double& r,
                                          const double& q,
                                          const double& sigma) {
    double sigma_sqr = pow(sigma, 2);
    double h1 = sqrt(pow(((r - q) / sigma_sqr), 2) + 2 * r / sigma_sqr);
    double price = (K / (h1 - 1.0)) * pow(((h1 - 1.0) / h1) * (S / K), h1);
    return price;
}
```

**Exercise 11.4.**

The price for an european put for a currency option is

\[
p = Ke^{-r(T-t)}N(-d_2) - Se^{-r(T-t)}N(-d_1)
\]

1. Implement this formula.

**11.5 Perpetual puts and calls**

A *perpetual* option is one with no maturity date, it is infinitely lived. Of course, only American perpetual options make any sense, European perpetual options would probably be hard to sell.\(^1\) For both puts and calls analytical formulas has been developed. We consider the price of an American call, and discuss the put in an exercise. Formula 11.5 gives the analytical solution.

\[
C^p = \frac{K}{h_1 - 1} \left( \frac{h_1 - 1}{h_1} \frac{S}{K} \right)^{h_1}
\]

where

\[
h_1 = \frac{1}{2} - \frac{r - q}{\sigma^2} + \sqrt{\left( \frac{r - q}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}}
\]

\(S\) is the current price of the underlying security, \(K\) is the exercise price, \(r\) is the risk free interest rate, \(q\) is the dividend yield and \(\sigma\) is the volatility of the underlying asset.

**Formula 11.5:** Price for a perpetual call option

\[^1\text{Such options would be like the classical April fools present, a perpetual zero coupon bond...}\]
Example
Price a perpetual call, given the following information $S = 50.0$, $K = 40.0$, $r = 0.05$, $q = 0.02$, $\sigma = 0.05$

C++ program:

```c++
double S=50.0; double K=40.0;
double r=0.05; double q=0.02;
double sigma=0.05;
double price = option_price_american_perpetual_call(S,K,r,q,sigma);
cout << " perpetual call price = " << price << endl;
```

Output from C++ program:

```
perpetual call price = 19.4767
```

Exercise 11.5.
The price for a perpetual american put is

$$P_p = \frac{K}{1 - h_2} \left( \frac{h_2 - 1}{h_2} S \right)^{h_2}$$

where

$$h_2 = \frac{1}{2} - \frac{r-q}{\sigma^2} - \sqrt{\left( \frac{r-q}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}}$$

1. Implement the calculation of this formula.

11.6 Readings

Hull (2011) and McDonald (2013) are general references. A first formulation of an analytical call price with dividends was in Roll (1977b). This had some errors, that were partially corrected in Geske (1979), before Whaley (1981) gave a final, correct formula. See Hull (2011) for a textbook summary. Black (1976) is the original development of the futures option. The original formulations of European foreign currency option prices are in Garman and Kohlhagen (1983) and Grabbe (1983). The price of a perpetual put was first shown in Merton (1973). For a perpetual call see McDonald and Siegel (1986). The notation for perpetual puts and calls follows the summary in (McDonald, 2013, pg. 393).
Chapter 12

Option pricing with binomial approximations

Contents

12.1 Introduction .................................................. 112
12.2 Pricing of options in the Black Scholes setting ................. 113
  12.2.1 European Options .................................... 114
  12.2.2 American Options ................................... 114
  12.2.3 Matlab implementation ................................. 116
12.3 How good is the binomial approximation? ....................... 119
  12.3.1 Estimating partials .................................... 120
12.4 Adjusting for payouts for the underlying ...................... 123
12.5 Pricing options on stocks paying dividends using a binomial approximation .......................... 124
  12.5.1 Checking for early exercise in the binomial model ........ 124
  12.5.2 Proportional dividends ................................ 124
  12.5.3 Discrete dividends .................................... 126
12.6 Option on futures ............................................ 128
12.7 Foreign Currency options .................................... 130
12.8 References ..................................................... 131

12.1 Introduction

We have shown binomial calculations given an up and down movement in chapter 8. However, binomial option pricing can also be viewed as an approximation to a continuous time distribution by judicious choice of the constants $u$ and $d$. To do so one has to ask: Is it possible to find a parametrization (choice of $u$ and $d$) of a binomial process
which has the same time series properties as a (continuous time) process with the same mean and volatility? There is actually any number of ways of constructing this, hence one uses one degree of freedom on imposing that the nodes reconnect, by imposing $u = \frac{1}{d}$.

To value an option using this approach, we specify the number $n$ of periods to split the time to maturity $(T - t)$ into, and then calculate the option using a binomial tree with that number of steps.

Given $S, X, r, \sigma, T$ and the number of periods $n$, calculate

$$
\Delta t = \frac{T - t}{n},
$$

$$
u = e^{\sigma \sqrt{\Delta t}},
$$

$$
d = e^{-\sigma \sqrt{\Delta t}}.
$$

We also redefine the “risk neutral probabilities”

$$
R = e^{r \Delta t},
$$

$$
q = \frac{R - d}{u - d}.
$$

To find the option price, will “roll backwards.” At node $t$, calculate the call price as a function of the two possible outcomes at time $t + 1$. For example, if there is one step,

$$
C_u = \max(0, S_u - X),
$$

$$
C_d = \max(0, S_d - X),
$$

find the call price at time 0 as

$$
C_0 = e^{-r}(qC_u + (1 - q)C_d).
$$

With more periods one will “roll backwards” as discussed in chapter 8.

12.2 Pricing of options in the Black Scholes setting

Consider options on underlying securities not paying dividend.

113
12.2.1 European Options

For European options, binomial trees are not that much used, since the Black Scholes model will give the correct answer, but it is useful to see the construction of the binomial tree without the checks for early exercise, which is the American case.

The computer algorithm for a binomial in the following merits some comments. There is only one vector of call prices, and one may think one needs two, one at time \( t \) and another at time \( t + 1 \). (Try to write down the way you would solve it before looking at the algorithm below.) But by using the fact that the branches reconnect, it is possible to get away with the algorithm below, using one less array. You may want to check how this works. It is also a useful way to make sure one understands binomial option pricing.

```cpp
#include <cmath>   // standard mathematical library
#include <algorithm>   // defining the max() operator
#include <vector>   // STL vector templates
using namespace std;

double option_price_call_european_binomial(const double & S, // spot price
                                           const double & K, // exercise price
                                           const double & r, // interest rate
                                           const double & sigma, // volatility
                                           const double & t, // time to maturity
                                           const int & steps){ // no steps in binomial tree
    double R = exp(r*t/steps); // interest rate for each step
    double Rinv = 1.0/R; // inverse of interest rate
    double u = exp(sigma*sqrt(t/steps)); // up movement
    double uu = u*u;
    double d = 1.0/u;
    double p_up = (R-d)/(u-d);
    double p_down = 1.0-p_up;
    vector<double> prices(steps+1); // price of underlying
    prices[0] = S*pow(d, steps); // fill in the endnodes.
    for (int i=1; i<=steps; ++i) prices[i] = uu*prices[i-1];
    vector<double> call_values(steps+1); // value of corresponding call
    for (int i=0; i<=steps; ++i) call_values[i] = max(0.0, (prices[i]-K)); // call payoffs at maturity
    for (int step=steps-1; step>=0; --step) {
        for (int i=0; i<=step; ++i) {
            call_values[i] = (p_up*call_values[i+1]+p_down*call_values[i])*Rinv;
        }
    }
    return call_values[0];
}
```

C++ Code 12.1: Option price for binomial European

12.2.2 American Options

An American option differs from an European option by the exercise possibility. An American option can be exercised at any time up to the maturity date, unlike the European option, which can only be exercised at maturity. In general, there is unfortunately no analytical solution to the American option problem, but in some cases it can be found. For example, for an American call option on non-dividend paying stock, the American price is the same as the European call.

It is in the case of American options, allowing for the possibility of early exercise, that binomial approximations are useful. At each node we calculate the value of the option as a function of the next periods prices, and then check for the value exercising of exercising the option now

C++ Code 12.2 illustrates the calculation of the price of an American call.
```cpp
#include <cmath>
#include <vector>
using namespace std;

double option_price_call_american_binomial(const double & S, 
                                        const double & K, 
                                        const double & r, 
                                        const double & sigma, 
                                        const double & t, 
                                        const int & steps) {
    double R = exp(r*(t/steps));
    double Rinv = 1.0/R;
    double u = exp(sigma*sqrt(t/steps));
    double d = 1.0/u;
    double p_up = (R-d)/(u-d);
    double p_down = 1.0-p_up;

    vector<double> prices(steps+1); // price of underlying
    prices[0] = S*pow(d, steps); // fill in the endnodes.
    double uu = u*u;
    for (int i=1; i<=steps; ++i) prices[i] = uu*prices[i-1];

    vector<double> call_values(steps+1); // value of corresponding call
    for (int i=0; i<=steps; ++i) call_values[i] = max(0.0, (prices[i]-K)); // call payoffs at maturity

    for (int step=steps-1; step>=0; --step) {
        for (int i=0; i<=step; ++i) {
            call_values[i] = (p_up*call_values[i+1]+p_down*call_values[i])*Rinv;
            prices[i] = d*prices[i+1];
            call_values[i] = max(call_values[i],prices[i]-K); // check for exercise
        }
    }
    return call_values[0];
}

C++ Code 12.2: Price of American call option using a binomial approximation

Actually, for this particular case, the american price will equal the european.
12.2.3 Matlab implementation

To illustrate differences between Matlab and C++, consider the two implementations in Matlab Code 12.1 and Matlab Code 12.2. The calculation of a put is more compact, with less loops, by some judicious use of array indexing.

```matlab
function c = bin_am_call( S, K, r, sigma, t, steps)
    R = exp(r*(t/steps));
    Rinv = 1.0/R;
    u = exp(sigma*sqrt(t/steps));
    d = 1.0/u;
    p_up = (R-d)/(u-d);
    p_down = 1.0- p_up;
    prices = zeros(steps+1);
    prices(1) = S*(d^steps);
    uu = u*u;
    for i=2:steps
        prices(i) = uu*prices(i-1);
    end
    call_values = max(0.0, (prices-K));
    for step=steps:-1:1
        for i=1:step+1
            call_values(i) = (p_up*call_values(i+1)+p_down*call_values(i))*Rinv;
            prices(i) = d*prices(i+1);
            call_values(i) = max(call_values(i),prices(i)-K);
        end
    end
    c = call_values(1);
end
```

Matlab Code 12.1: Price of American call using a binomial approximation
function p = bin_am_put( S, K, r, sigma, t, steps)
    R = exp(r*(t/steps));
    Rinv = 1.0/R;
    u = exp(sigma*sqrt(t/steps));
    d = 1.0/u;
    p_up = (R-d)/(u-d);
    p_down = 1.0-p_up;

    prices = zeros(steps+1,1);
    prices(1) = S*(d^steps);
    uu = u*u;
    for i=2:steps+1
        prices(i) = uu*prices(i-1);
    end
    values = max(0.0, (K-prices));
    for step=steps:-1:1
        values = Rinv * ( p_up*values(2:step+1) + p_down*values(1:step) );
        prices = u*prices(1:step);
        values = max(values,K-prices);
    end
    p=values(1);
end

Matlab Code 12.2: Price of American put using a binomial approximation
Example
You are given the following information about an option: $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$ and time to maturity is 1 year. Price American calls and puts using binomial approximations with 100 steps.

C++ program:

```cpp
double S = 100.0; double K = 100.0;
double r = 0.1; double sigma = 0.25;
double time=1.0; int no_steps = 100;

cout << " european call= " << option_price_call_european_binomial(S,K,r,sigma,time,no_steps) << endl;
cout << " american call= " << option_price_call_american_binomial(S,K,r,sigma,time,no_steps) << endl;
cout << " american put = " << option_price_put_american_binomial(S,K,r,sigma,time,no_steps) << endl;
```

Output from C++ program:

- European call: 14.9505
- American call: 14.9505
- American put: 6.54691

Matlab program:

```matlab
S=100; K=100; r=0.1; sigma=0.25; time=1; no_steps=100;
C = bin_am_call(S,K,r,sigma,time,no_steps)
P = bin_am_put(S,K,r,sigma,time,no_steps)
```

Output from Matlab program:

- $C = 14.951$
- $P = 6.5469$
12.3 How good is the binomial approximation?

To illustrate the behaviour of the binomial approximation figure 12.1 plots a comparison with the binomial approximation as a function of \( n \), the number of steps in the binomial approximation, and the true (Black Scholes) value of the option. Note the “sawtooth” pattern, the binomial approximation jumps back and forth around the true value for small values of \( n \), but rapidly moves towards the Black Scholes value as the \( n \) increases.

Figure 12.1: Illustrating convergence of binomial to Black Scholes
12.3.1 Estimating partials.

It is always necessary to calculate the partial derivatives as well as the option price. Let us start with Delta, the derivative of the option price with respect to the underlying. The binomial methods gives us ways to approximate these as well. How to find them in the binomial case are described in Hull (2011).

The implementation in C++ Code 12.3 is for the non–dividend case.

```cpp
#include <cmath>
#include <algorithm>
#include <vector>

using namespace std;

double option_price_delta_american_call_binomial(const double &S,
const double &K,
const double &r,
const double &sigma,
const double &t,
const int &no_steps){  // steps in binomial
    double R = exp(r*(t/no_steps));
    double Rinv = 1.0/R;
    double u = exp(sigma*sqrt(t/no_steps));
    double d = 1.0/u;
    double uu = u*u;
    double pUp = (R-Rinv) / (u-d);
    double pDown = 1.0 - pUp;

    vector<double> prices(no_steps+1);
    prices[0] = S*pow(d, no_steps);
    for (int i=1; i<=no_steps; ++i) prices[i] = uu*prices[i-1];

    vector<double> call_values(no_steps+1);
    for (int i=0; i<=no_steps; ++i) call_values[i] = max(0.0, (prices[i]-K));

    for (int CurrStep=no_steps-1; CurrStep>=1; --CurrStep) {
        for (int i=0; i<=CurrStep; ++i) {
            prices[i] = d*prices[i+1];
            call_values[i] = (pDown*call_values[i]+pUp*call_values[i+1])*Rinv;
            call_values[i] = max(call_values[i], prices[i]-K);  // check for exercise
        }
    }
    double delta = (call_values[1]-call_values[0])/(S*u-S*d);
    return delta;
}
```

C++ Code 12.3: Delta

Calculation of all the derivatives are shown in C++ Code 12.4.
```cpp
#include <cmath>
#include <algorithm>
#include "fin_recipes.h"

void option_price_partials_american_call_binomial(const double& S, // spot price
    const double& K, // Exercise price,
    const double& r, // interest rate
    const double& sigma, // volatility
    const double& t, // time to maturity
    const int& no_steps, // steps in binomial
    double& delta, // partial wrt S
double& gamma, // second partial wrt S
double& theta, // partial wrt time
double& vega, // partial wrt sigma
double& rho){ // partial wrt r

    vector<double> prices(no_steps+1);
    vector<double> call_values(no_steps+1);
    double delta_t = (t/no_steps);
    double R = exp(r*delta_t);
    double Rinv = 1.0/R;
    double u = exp(sigma*sqrt(delta_t));
    double d = 1.0/u;
    double uu= u*u;
    double pUp = (R-d)/(u-d);
    double pDown = 1.0 - pUp;
    prices[0] = S*exp(d*no_steps);
    for (int i=1; i<=no_steps; ++i) prices[i] = uu*prices[i-1];
    for (int i=0; i<=no_steps; ++i) call_values[i] = max(0.0, (prices[i]-K));
    for (int CurrStep=0; CurrStep<=no_steps-1; CurrStep++)
    for (int i=0; i<=CurrStep; ++i) {
        prices[i] = d*prices[i+1];
        call_values[i] = (pDown*call_values[i]+pUp*call_values[i+1])*Rinv;
        call_values[i] = max(call_values[i], prices[i]-K); // check for exercise
    }
    double f22 = call_values[2];
    double f21 = call_values[1];
    double f20 = call_values[0];
    for (int i=0;i<=1;i++) {
        prices[i] = d*prices[i+1];
        call_values[i] = (pDown*call_values[i]+pUp*call_values[i+1])*Rinv;
        call_values[i] = max(call_values[i], prices[i]-K); // check for exercise
    }
    double f11 = call_values[1];
    double f10 = call_values[0];
    prices[0] = d*prices[1];
    call_values[0] = max(call_values[0], S-K); // check for exercise on first date
    double f00 = call_values[0];
    delta = (f11-f10)/(S*uu-S*d);
    double h = 0.5 * S * ( uu - d*d);
    gamma = ((f22-f21)/(S*(uu-1)) = (f21-f20)/(S*(1-d*d)) ) / h;
    theta = (f21-f00) / (2*delta_t);
    double diff = 0.02;
    double tmp_sigma = sigma+d*diff;
    double tmp_prices = option_price_call_american_binomial(S,K,r,tmp_sigma,t,0,0);
    vega = (tmp_prices-f00)/diff;
    diff = 0.05;
    double tmp_r = r+diff;
    tmp_prices = option_price_call_american_binomial(S,K,tmp_r,sigma,t,0,0);
    rho = (tmp_prices-f00)/diff;
);
```

C++ Code 12.4: Hedge parameters
Example
Given the following information: $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$ and time to maturity is 1 year, use 100 steps in the binomial approximation.

1. Estimate all the “greeks” for the option: $\Delta$, gamma, theta, vega and rho.

```
C++ program:

double S = 100.0; double K = 100.0;
double r = 0.1; double sigma = 0.25;
double time=1.0; int no_steps = 100;
double delta, gamma, theta, vega, rho;
option_pricepartials_american_call_binomial(S,K,r,sigma,time,no_steps,
delta, gamma, theta, vega, rho);
cout << " Call price partials " << endl;
cout << " delta = " << delta << endl;
cout << " gamma = " << gamma << endl;
cout << " theta = " << theta << endl;
cout << " vega = " << vega << endl;
cout << " rho = " << rho << endl;
```

Output from C++ program:

```
Call price partials
delta = 0.699792
gamma = 0.0140407
theta = -9.89067
vega = 34.8536
rho = 56.9652
```

Exercise 12.1.
Consider an American call option on non-dividend paying stock, where $S = 100$, $K = 100$, $\sigma = 0.2$, $(T-t) = 1$ and $r = 0.1$.

1. Calculate the price of this option using Black Scholes
2. Calculate the price using a binomial approximation, using 10, 100 and 1000 steps in the approximation.
3. Discuss sources of differences in the estimated prices.
12.4 Adjusting for payouts for the underlying

The simplest case of a payout is the similar one to the one we saw in the Black Scholes case, a continuous payout of $y$. This $y$ needs to be brought into the binomial framework. Here we use it to adjust the probability:

\[
\Delta t = \sqrt{(T - t)/n}
\]

\[
u = e^{\sigma \Delta t}
\]

\[
d = \frac{1}{u}
\]

\[
p_u = \frac{e^{(r-y)\Delta t} - d}{u - d}
\]

\[p_d = 1 - p_u \]

For example, the last node

\[
C_0 = e^{-r \Delta t}(p_u C_u + (1 - p_u)C_d)
\]

```cpp
#include <cmath>        // standard mathematical library
#include <algorithm>    // defines the max() operator
#include <vector>       // STL vector templates
using namespace std;

double option_price_call_american_binomial( const double& S, // spot price
c                                             const double& K, // exercise price
c                                             const double& r, // interest rate
c                                             const double& y, // continuous payout
c                                             const double& sigma, // volatility
c                                             const double& t, // time to maturity
                                             const int& steps) { // no steps in binomial tree
double R = exp(r*(t/steps));   // interest rate for each step
double Rinv = 1.0/R;            // inverse of interest rate
double u = exp(sigma*sqrt(t/steps)); // up movement
double uu = u*u;
double d = 1.0/u;
double p_up = (exp((r-y)*(t/steps))-d)/(u-d);
double p_down = 1.0-p_up;
vector<double> prices(steps+1); // price of underlying
prices[0] = S*pow(d, steps);
for (int i=1; i<=steps; ++i) prices[i] = uu*prices[i-1]; // fill in the endnodes.

vector<double> call_values(steps+1); // value of corresponding call
for (int i=0; i<=steps; ++i) call_values[i] = max(0.0, (prices[i]-K)); // call payoffs at maturity

for (int step=steps-1; step>=0; --step) {
    for (int i=0; i<=step; ++i) {
        call_values[i] = (p_up*call_values[i+1]+p_down*call_values[i])*Rinv;
        prices[i] = d*prices[i+1];
        call_values[i] = max(call_values[i],prices[i]-K); // check for exercise
    }
}
return call_values[0];
```

C++ Code 12.5: Binomial option price with continuous payout
12.5 Pricing options on stocks paying dividends using a binomial approximation

12.5.1 Checking for early exercise in the binomial model.

If the underlying asset is a stock paying dividends during the maturity of the option, the terms of the option is not adjusted to reflect this cash payment, which means that the option value will reflect the dividend payments.

In the binomial model, the adjustment for dividends depend on whether the dividends are discrete or proportional.

12.5.2 Proportional dividends.

For proportional dividends, we simply multiply with an adjustment factor the stock prices at the ex-dividend date, the nodes in the binomial tree will “link up” again, and we can use the same “rolling back” procedure.
```cpp
#include <cmath>
#include <algorithm>
#include <vector>
#include "fin_recipes.h"
#include <iostream>

double option_price_call_american_proportional_dividends_binomial(const double& S,
    const double& K,
    const double& r,
    const double& sigma,
    const double& time,
    const int& no_steps,
    const vector<double>& dividend_times,
    const vector<double>& dividend_yields) {

    // note that the last dividend date should be before the expiry date, problems if dividend at terminal node
    if (no_dividends == 0) {
        return option_price_call_american_binomial(S,K,r,sigma,time,no_steps); // price w/o dividends
    }
    double delta_t = time/no_steps;
    double R = exp(r*delta_t);
    double Rinv = 1.0/R;
    double u = exp(sigma*sqrt(delta_t));
    double uu = u*u;
    double d = 1.0/u;
    double pUp = (R-d)/(u-d);
    double pDown = 1.0 - pUp;
    vector<int> dividend_steps(no_dividends); // when dividends are paid
    for (int i=0; i<no_dividends; ++i) {
        dividend_steps[i] = (int)(dividend_times[i]/time*no_steps);
    }
    vector<double> prices(no_steps+1);
    vector<double> call_prices(no_steps+1);
    prices[0] = S*pow(d, no_steps); // adjust downward terminal prices by dividends
    for (int i=0; i<no_dividends; ++i) { prices[i] = prices[i]*(1.0 - dividend_yields[i]); }
    for (int i=1; i<no_steps; ++i) { prices[i] = uu*prices[i-1]; }
    for (int i=0; i<no_steps; ++i) call_prices[i] = max(0.0, (prices[i]-K));

    for (int step=no_steps-1; step>=0; --step) {
        for (int i=0;i<no_dividends; ++i) { // check whether dividend paid
            if (step==dividend_steps[i]) {
                for (int j=0;j<=(step+1);++j) {
                    prices[i] = prices[i]*(1.0/(1.0-dividend_yields[i]));
                }
            }
        }
        for (int i=0; i<step; ++i) {
            call_prices[i] = (pDown*call_prices[i] + pUp*call_prices[i+1])*Rinv;
            prices[i] = d*prices[i+1];
            call_prices[i] = max(call_prices[i], prices[i]-K); // check for exercise
        }
    }
    return call_prices[0];
}
```

C++ Code 12.6: Binomial option price of stock option where stock pays proportional dividends
12.5.3 Discrete dividends

The problem is when the dividends are constant dollar amounts. In that case the nodes of the binomial tree do not “link up,” and the number of branches increases dramatically, which means that the time to do the calculation is increased.

The algorithm presented in C++ Code 12.7 implements this case, with no linkup, by constructing a binomial tree up to the ex-dividend date, and then, at the terminal nodes of that tree, call itself with one less dividend payment, and time to maturity the time remaining at the ex-dividend date. Doing that calculates the value of the option at the ex-dividend date, which is then compared to the value of exercising just before the ex-dividend date. It is a instructive example of using recursion in simplifying calculations, but as with most recursive solutions, it has a cost in computing time. For large binomial trees and several dividends this procedure is costly in computing time.
```cpp
#include <cmath>
#include <vector>
#include "fin_recipes.h"
#include <iostream>

double option_price_call_american_discrete_dividends_binomial(const double& S, const double& K, const double& r, const double& sigma, const double& t, const int& steps, const int& dividend, const double& dividend_times, const double& dividend_amounts) {
    int no_dividends = int(dividend_times.size());
    if (no_dividends==0) return option_price_call_american_binomial(S, K, r, sigma, t, steps); // just do regular
    int steps_before_dividend = (int)(dividend_times[0]/t*steps);
    const double R = exp(st*(t/steps));
    const double Rinv = 1.0/R;
    const double u = exp(sigma*sqrt(t/steps));
    const double d = 1.0/u;
    const double pUp = (R-d)/(u-d);
    const double pDown = 1.0 - pUp;
    double dividend_amount = dividend_amounts[0];
    vector<double> tmp_dividend_times(no_dividends-1); // temporaries with vector<double> tmp_dividend_amounts(no_dividends-1); // one less dividend
    for (int i=0; i<no_dividends-1; ++i) {
        tmp_dividend_amounts[i] = dividend_amounts[i+1];
        tmp_dividend_times[i] = dividend_times[i+1] - dividend_times[i];
    }
    vector<double> prices(steps_before_dividend+1);
    vector<double> call_value(steps_before_dividend+1);
    prices[0] = S*exp(d*steps/dBeforeDividend);
    for (int i=1; i<=steps_before_dividend; ++i) prices[i] = u*u*prices[i-1];
    for (int i=0; i<=steps_before_dividend; ++i) {
        double value_alive = option_price_call_american_discrete_dividends_binomial(prices[i]-dividend_amount,K, r, sigma, t-dividend_times[i], // time after first dividend steps-steps_before_dividend, tmp_dividend_times, tmp_dividend_amounts);
        call_values[i] = max(value_alive, (prices[i]-K)); // compare to exercising now
    }
    for (int step=steps_before_dividend-1; step>=0; --step) {
        for (int i=0; i<=step; ++i) {
            prices[i] = d*prices[i-1];
            call_values[i] = (pDown*call_values[i]+pUp*call_values[i+1])*Rinv;
            call_values[i] = max(call_values[i], prices[i]-K);
        }
    }
    return call_values[0];
}

C++ Code 12.7: Binomial option price of stock option where stock pays discrete dividends
```
Example
Given the following information $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$ and time to maturity is 1 year, Calculate option price with two different assumptions about dividends:

1. Continuous payout $d = 0.02$.
2. Discrete payout, $d = 0.025$ at times 0.25 and 0.75.

C++ program:
```cpp
double S = 100.0; double K = 100.0;
double r = 0.10; double sigma = 0.25;
double time=1.0;
int no_steps = 100;
double d=0.02;
cout << " call price with continuous dividend payout = " << option_price_call_american_binomial(S,K,r,sigma,time,no_steps) << endl;
vector<double> dividend_times; dividend_times.push_back(0.25); dividend_yields.push_back(0.025);
dividend_times.push_back(0.75); dividend_yields.push_back(0.025);
cout << " call price with proportional dividend yields at discrete dates = " << endl;

vector<double> dividend_amounts; dividend_amounts.push_back(2.5); dividend_amounts.push_back(2.5);
cout << " call price with proportional dividend amounts at discrete dates = " << option_price_call_american_discrete_dividends_binomial(S,K,r,sigma,time,no_steps)
```

Output from C++ program:
```
call price with continuous dividend payout = 13.5926
call price with proportional dividend yields at discrete dates = 11.8604
```

12.6 Option on futures
For American options, because of the feasibility of early exercise, the binomial model is used to approximate the option value for both puts and calls.

Example
$F = 50.0$, $K = 45.0$, $r = 0.08$, $\sigma = 0.2$, $time=0.5$, no steps=100; Price the futures option

C++ program:
```cpp
double F = 50.0; double K = 45.0;
double r = 0.08; double sigma = 0.2;
double time=0.5;
int no_steps=100;
cout << " european futures call option = " << futures_option_price_call_american_binomial(F,K,r,sigma,time,no_steps) << endl;
```

Output from C++ program:
```
european futures call option = 5.74254
```
```cpp
#include <cmath>
#include <algorithm>
#include <vector>

using namespace std;

double futures_option_price_call_american_binomial(const double& F, // price futures contract
c                        const double& K, // exercise price
c                        const double& r, // interest rate
cc                       const double& sigma, // volatility
cc                       const double& time, // time to maturity
cc                       const int& no_steps) { // number of steps

    vector<double> futures_prices(no_steps + 1);
    vector<double> call_values(no_steps + 1);
    double t_delta = time / no_steps;
    double Rinv = exp(-r * t_delta);
    double u = exp(sigma * sqrt(t_delta));
    double d = 1.0 / u;
    double uu = u * u;
    double pUp = (1 - d) / (u - d); // note how probability is calculated
    double pDown = 1.0 - pUp;
    futures_prices[0] = F * pow(d, no_steps);
    int i;
    for (i = 1; i <= no_steps; ++i) futures_prices[i] = uu * futures_prices[i - 1]; // terminal tree nodes
    for (i = 0; i <= no_steps; ++i) call_values[i] = max(0.0, futures_prices[i] - K);
    for (int step = no_steps - 1; step >= 0; --step) {
        for (i = 0; i <= step; ++i) {
            futures_prices[i] = d * futures_prices[i + 1];
            call_values[i] = (pDown * call_values[i] + pUp * call_values[i + 1]) * Rinv;
            call_values[i] = max(call_values[i], futures_prices[i] - K); // check for exercise
        }
    }
    return call_values[0];
}
```

C++ Code 12.8: Pricing an american call on an option on futures using a binomial approximation
12.7 Foreign Currency options

For American options, the usual method is approximation using binomial trees, checking for early exercise due to the interest rate differential.

```cpp
#include <cmath>
#include <algorithm>
#include <vector>
using namespace std;

double currency_option_price_call_american_binomial(const double& S,
    const double& K,
    const double& r,
    const double& r_f,
    const double& sigma,
    const double& time,
    const int& no_steps) {

    vector<double> exchange_rates(no_steps+1);
    vector<double> call_values(no_steps+1);
    double t_delta= time/no_steps;
    double Rinv = exp(-r*(t_delta));
    double u = exp(sigma*sqrt(t_delta));
    double d = 1.0/u;
    double uu = u*u;
    double pUp = (exp((r-r_f)*t_delta)-d)/(u-d); // adjust for foreign int.rate
    double pDown = 1.0 - pUp;
    exchange_rates[0] = S*pow(d, no_steps);
    int i;
    for (i=1; i<=no_steps; ++i) {
        exchange_rates[i] = uu*exchange_rates[i-1]; // terminal tree nodes
    }
    for (i=0; i<=no_steps; ++i) call_values[i] = max(0.0, (exchange_rates[i]-K));
    for (int step=no_steps-1; step>=0; --step) {
        for (i=0; i<=step; ++i) {
            exchange_rates[i] = d*exchange_rates[i+1];
            call_values[i] = (pDown*call_values[i]+pUp*call_values[i+1])*Rinv;
            call_values[i] = max(call_values[i], exchange_rates[i]-K); // check for exercise
        }
    }
    return call_values[0];
}
```

**C++ Code 12.9:** Pricing an american call on an option on currency using a binomial approximation

**Example**

Price a futures currency option with the following information: $S = 50$, $K = 52$, $r = 0.08$, $r_f = 0.05$, $\sigma = 0.2$, $\text{time}=0.5$, number of steps = 100.

```cpp
double S = 50.0;  
double K = 52.0;  
double r = 0.08;  
double rf = 0.05;  
double sigma = 0.2;  
double time=0.5;  
int no_steps = 100;  
cout << "european currency option call = " << currency_option_price_call_american_binomial(S,K,rf,sigma,time,no_steps) << endl;
```

**Output from C++ program:**

```plaintext
european currency option call = 2.23129
```
12.8 References

The original source for binomial option pricing was the paper by Cox et al. (1979). Textbook discussions are in Cox and Rubinstein (1985), Bossaerts and Ødegaard (2001) and Hull (2011).
Chapter 13

Finite Differences

Contents

13.1 Explicit Finite differences .................................................. 132
13.2 European Options ......................................................... 132
13.3 American Options .......................................................... 134
13.4 Implicit finite differences ................................................... 137
13.5 An example matrix class .................................................... 137
13.6 Finite Differences ............................................................ 137
13.7 American Options ............................................................ 137
13.8 European Options ............................................................ 140
13.9 References ................................................................. 141

13.1 Explicit Finite differences

The method of choice for any engineer given a differential equation to solve is to numerically approximate it using a finite difference scheme, which is to approximate the continuous differential equation with a discrete difference equation, and solve this difference equation.

13.2 European Options.

For European options we do not need to use the finite difference scheme, but we show how one would find the European price for comparison purposes. We show the case of an explicit finite difference scheme in C++ Code 13.1. A problem with the explicit version is that it may not converge for certain combinations of inputs.
#include <cmath>
#include <vector>

using namespace std;

double option_price_put_european_finite_diff_explicit(const double& S,
const double& X,
const double& r,
const double& sigma,
const double& time,
const int& no_S_steps,
const int& no_t_steps) {
    double sigma_sqr = pow(sigma,2);
    int M=no_S_steps; if ((no_S_steps%2)==1) { ++M; } // need no_S_steps to be even:
    double delta_S = 2.0*S/M;
    vector<double> S_values(M+1);
    for (unsigned m=0;m<=M;m++) { S_values[m] = m*delta_S; }
    int N=no_t_steps;
    double delta_t = time/N;

    vector<double> a(M);
    vector<double> b(M);
    vector<double> c(M);
    double r1=1.0/(1.0+r*delta_t);
    double r2=delta_t/(1.0+r*delta_t);
    for (unsigned int j=1;j<M;j++){
        a[j] = r2*0.5*j*(-r+sigma_sqr); 
        b[j] = r1*(1.0-sigma_sqr)*delta_t;
        c[j] = r2*0.5*j*(r+sigma_sqr);
    }

    vector<double> f_next(M+1);
    for (unsigned m=0;m<=M;++m) { f_next[m]=max(0.0,X-S_values[m]); }
    double f[M+1];
    for (int t=N-1;t>=0; --t) {
        f[0]=X;
        for (unsigned m=1;m<=M;++m) {
            if(m==0) f[m]=a[m]*f_next[m]+b[m]*f_next[m-1]+c[m]*f_next[m+1];
            if(m==M) f[m] = 0;
            if(m==0) f[m]=max(0.0,m+M);
            f[m+1] = f[m];
        }
    }
    return f[M/2];
}

C++ Code 13.1: Explicit finite differences calculation of european put option
13.3 American Options.

We now compare the American versions of the same algorithms, the only difference being the check for exercise at each point. C++ Code 13.2 shows the C++ code for an American put option and Matlab Code 13.1 shows the same implemented in Matlab.

```cpp
#include <cmath>
#include <algorithm>
#include <vector>
using namespace std;

double option_price_put_american_finite_diff_explicit(const double& S,
  const double& K,
  const double& r,
  const double& sigma,
  const double& time,
  const int& no_S_steps,
  const int& no_t_steps) {

double sigma_sqr = sigma*sigma;
int M=no_S_steps+(no_S_steps%2); // need no_S_steps to be even:
double delta_S = 2.0*S/M;
vector<double> S_values(M+1);
for (int m=0;m<=M;m++) { S_values[m] = m*delta_S; }
int N=no_t_steps;
double delta_t = time/N;

vector<double> a(M);
vector<double> b(M);
vector<double> c(M);
double r1=1.0/(1.0+r*delta_t);
double r2=delta_t/(1.0+r*delta_t);
for (int j=1;j<=M;j++){
  a[j] = r2*(0.5)*(r+sigma_sqr*j);
  b[j] = r1*(1.0-sigma_sqr*j)*delta_t;
  c[j] = r2*(0.5)*(r+sigma_sqr*j);
};
vector<double> f_next(M+1);
for (int m=0;m<=M+m) { f_next[m]=max(0.0,K-S_values[m]); }
vector<double> f(M+1);

for (int t=N-1; t>=0; t--) {
  f[0]=K;
  for (int m=1;m<=M;++m) {
    f[m]=a[m]*f_next[m-1]+b[m]*f_next[m]+c[m]*f_next[m+1];
    if (f[m] == max(f[m],K-S_values[m])); // check for exercise
  }
  f[M] = 0;
  for (int m=0;m<=M;++m) { f_next[m] = f[m]; }
};
return f[M/2];
}
```

C++ Code 13.2: Explicit finite differences calculation of American put option
function \( P = \text{findiff\_exp\_am\_put}(S, K, r, \sigma, \text{time}, \text{no\_S\_steps}, \text{no\_t\_steps}) \)

\[
\begin{align*}
\text{sigma\_sqr} & = \sigma^2; \\
M & = \text{no\_S\_steps} + \text{rem(\text{no\_S\_steps},2)}; \quad \# \text{need \text{no\_S\_steps} to be even:} \\
\text{delta\_S} & = 2S/M; \\
\text{S\_values} & = \text{delta\_S} \times (0:M)'; \\
N & = \text{no\_t\_steps}; \\
\text{delta\_t} & = \text{time}/N;
\end{align*}
\]

\[
\begin{align*}
r1 & = 1/(1+r*\text{delta\_t}); \\
r2 & = \text{delta\_t}/(1+r*\text{delta\_t}); \\
a & = \text{zeros}(M-1,1); \\
b & = \text{zeros}(M-1,1); \\
c & = \text{zeros}(M-1,1); \\
\text{for } j=1:M-1 \\
\quad a(j) & = r2*0.5*j*(-r+\text{sigma\_sqr}); \\
\quad b(j) & = r1*1.0-\text{sigma\_sqr}^2*\text{delta\_t}; \\
\quad c(j) & = r2*0.5*j*(r+\text{sigma\_sqr}); \\
\text{endfor}
\]

\[
\begin{align*}
\text{f\_next} & = \max(0,K-\text{S\_values}); \\
\text{for } t=N-1:-1:0 \\
\quad f & = [ K, a.*\text{f\_next}(1:M-1)+b.*\text{f\_next}(2:M)+c.*\text{f\_next}(3:M+1); 0]; \\
\quad f & = \max(f,K-S\_values); \\
\quad \text{f\_next} & = f; \\
\text{endfor}
\]

\[
P = f(1+M/2);
\]

Matlab Code 13.1: Explicit finite differences calculation of American put option
Example
Given the following parameters: \( S = 50, K = 50, r = 10\% \) and \( \sigma = 0.4 \). The time to maturity is 0.4167.
Price European and American put option prices using finite differences with 20 steps in the \( S \) dimension and 11 steps in the time dimension.

\[
\text{C++ program:}
\]
\[
\begin{align*}
\text{double } S &= 50.0; \\
\text{double } K &= 50.0; \\
\text{double } r &= 0.1; \\
\text{double } \sigma &= 0.4; \\
\text{double } \text{time} &= 0.4167; \\
\text{int } \text{no}_S\text{steps} &= 20; \\
\text{int } \text{no}_t\text{steps} &= 11; \\
\text{cout } \ll " \text{explicit finite differences, european put price } = " ; \\
\text{cout } \ll \text{option\_price\_put\_european\_finite\_diff\_explicit}(S,K,r,\sigma,\text{time},\text{no}_S\text{steps},\text{no}_t\text{steps}) \\
\ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll \ll
\]
\[
\begin{align*}
\text{Output from C++ program:}
\end{align*}
\]
\[
\begin{align*}
\text{explicit finite differences, european put price } &= 4.03667; \\
\text{explicit finite differences, american put price } &= 4.25085.
\end{align*}
\]

Readings  Brennan and Schwartz (1978) is one of the first finance applications of finite differences. Section 14.7 of Hull (1993) has a short introduction to finite differences. Wilmott, Dewynne, and Howison (1994) is an exhaustive source on option pricing from the perspective of solving partial differential equations.
13.4 Implicit finite differences

What really distinguishes C++ from standard C is the ability to extend the language by creating classes and collecting these classes into libraries. A library is a collection of classes and routines for one particular purpose. We have already seen this idea when creating the date and term_structure classes. However, one should not necessarily always go ahead and create such classes from scratch. It is just as well to use somebody else's class, as long as it is correct and well documented and fulfills a particular purpose.

13.5 An example matrix class

Use Newmat as an example matrix class.

13.6 Finite Differences

We use the case of implicit finite difference calculations to illustrate matrix calculations in action. The method of choice for any engineer given a differential equation to solve is to numerically approximate it using a finite difference scheme, which is to approximate the continuous differential equation with a discrete difference equation, and solve this difference equation.

In the following we implement implicit finite differences. Explicit finite differences was discussed earlier, we postponed the implicit case to now because it is much simplified by a matrix library.

13.7 American Options

Let us first look at how this pricing is implemented in Matlab. Matlab Code 13.2 shows the implementation. Implementation of the same calculation in C++ Code 13.3 using the Newmat library and C++ Code 13.4 using IT++. 

```
function P = findiff_imp_am_put(S,K,r,sigma,timesteps,ntimesteps)
    sigma_sqr = sigma^2;
    M = no_S_steps + rem(no_S_steps,2); # need no_S_steps to be even:
    delta_S = 2*S/double(M);
    S_values = delta_S* (1:M+1);
    N = no_t_steps;
    delta_t = time/N;
    A(1,1)=1.0;
    for j=2:M
        A(j,j-1) = 0.5*delta_t*(r-sigma_sqr);
        A(j,j) = 1.0 + delta_t*(r+sigma_sqr);
        A(j,j+1) = 0.5*delta_t*(-r-sigma_sqr);
    endfor
    A(M+1,M+1)=1.0;
    B = max(0,K-S_values);
    F = inv(A)*B;
    for t=N-1:-1:1
        B = F;
        F = inv(A)*B;
        F=max(F,K-S_values);
    endfor
    P = F(M/2);
endfunction
```

Matlab Code 13.2: Calculation of price of American put using implicit finite differences

137
#include <cmath>
#include "newmat.h" // definitions for newmat matrix library
using namespace NEWMAT;
#include <vector>
#include <algorithm>
using namespace std;

double option_price_put_american_finite_diff_implicit(const double& S,
    const double& K,
    const double& r,
    const double& sigma,
    const double& time,
    const int& no_S_steps,
    const int& no_t_steps) {
    double sigma_sqr = sigma*sigma;
    int M=no_S_steps + (no_S_steps%2); // need no_S_steps to be even:
    // int M=no_S_steps; if ((no_S_steps%2)==1) { ++M; }; // need no_S_steps to be even:
    double delta_S = 2.0*S/double(M);
    double S_values[M+1];
    for (int m=0;m<=M;m++) { S_values[m] = m*delta_S; }
    int N=no_t_steps;
    double delta_t = time/N;

    BandMatrix A(M+1,1,1);
    A.element(0,0) = 1.0;
    for (int j=1;j<M;j++) {
        A.element(j,j-1) = 0.5*j*delta_t*(r-sigma_sqr); // a[j]
        A.element(j,j) = 1.0 + delta_t*(r+sigma_sqr); // b[j];
        A.element(j,j+1) = 0.5*j*delta_t*(-r-sigma_sqr); // c[j];
    }
    A.element(M,M)=1.0;
    ColumnVector B(M+1);
    for (int m=0;m<=M;m++) { B.element(m) = max(0.0,K-S_values[m]); }
    ColumnVector F=A.i()*B;
    for (int t=N-1;t>=0;--t) {
        B = F;
        F = A.i()*B;
        for (int m=1;m<M;++m) { // now check for exercise
            F.element(m) = max(F.element(m), K-S_values[m]);
        }
    }
    return F.element(M/2);
}

C++ Code 13.3: Calculation of price of American put using implicit finite differences with the Newmat matrix library
double option_price_put_american_finite_diff_implicit(ITPP(const double& S,  
const double& K,  
const double& r,  
const double& sigma,  
const double& time,  
const int& no_S_steps,  
const int& no_t_steps) {  
    double sigma_sqr = sigma*sigma;  
    int M=no_S_steps + (no_S_steps%2); // need no_S_steps to be even:  
    double delta_S = 2.0*S/double(M);  
    double S_values[M+1];  
    for (int m=0;m<=M;++m) { S_values[m] = m*delta_S; }  
    int N=no_t_steps;  
    double delta_t = time/N;  
    mat A(M+1,M+1);  
    A.zeros();  
    A(0,0) = 1.0;  
    for (int j=1;j<M;++j) {  
        A(j,j-1) = 0.5*delta_t*(r-sigma_sqr); // a[j]  
        A(j,j) = 1.0 + delta_t*(r+sigma_sqr); // b[j];  
        A(j,j+1) = 0.5*delta_t*(-r-sigma_sqr); // c[j];  
    }  
    A(M,M)=1.0;  
    vec B(M+1),  
    for (int m=0;m<=M;++m) { B(m) = max(0.0,K-S_values[m]); }  
    mat InvA = inv(A);  
    vec F=InvA*B;  
    for(int t=N-1;t>=0;--t) {  
        B = F;  
        F = InvA*B;  
        for (int m=1;m<M;++m) { // now check for exercise  
            F(m) = max(F(m), K-S_values[m]);  
        }  
    }  
    return F(M/2);  
};

C++ Code 13.4: Calculation of price of American put using implicit finite differences with the IT++ matrix library
13.8 European Options

For European options we do not need to use the finite difference scheme, but for comparison purposes C++ Code 13.5 show how one would find the European price.

C++ Code 13.5: Calculation of price of European put using implicit finite differences

```cpp
#include "newmat.h" // definitions for newmat matrix library
using namespace NEWMAT;
#include <cmath>
#include <vector>
#include <algorithm>
using namespace std;

double option_price_put_european_finite_diff_implicit(const double & S,
const double & K,
const double & r,
const double & sigma,
const double & time,
const int & no_S_steps,
const int & no_t_steps) {

double sigma_sqr = sigma*sigma;
int M=no_S_steps + (no_S_steps%2); // need no_S_steps to be even:
// int M=no_S_steps; if ((no_S_steps%2)==1) { ++M; }; // need no_S_steps to be even:
double delta_S = 2.0*S/M;
vector<double> S_values(M+1);
for (int m=0;m<=M;m++) { S_values[m] = m*delta_S; }
int N=no_t_steps;
double delta_t = time/N;

BandMatrix A(M+1,1,1); A=0.0;
A.element(0,0) = 1.0;
for (int j=1;j<M;++j) {
    A.element(j,j-1) = 0.5*delta_t*(r-sigma_sqr);
    A.element(j,j) = 1.0 + delta_t*(r+sigma_sqr);
    A.element(j,j+1) = 0.5*delta_t*(-r-sigma_sqr);
}
A.element(M,M)=1.0;
ColumnVector B(M+1);
for (int m=0;m<=M;++m) { B.element(m) = max(0.0,K-S_values[m]); }
ColumnVector P=A.i()*B;
for(int t=N-1;t>0;--t) {
    B = P;
    P = A.i()*B;
}
return P.element(M/2);
}
```

Exercise 13.1.

The routines above uses direct multiplication with the inverse of $A$. Are there alternative, numerically more stable ways of doing it?

Example

Given the parameters $S = 50$, $K = 50$, $r = 0.1$, $\sigma = 0.4$, $t = 0.5$, no $S$ steps=200; no $t$ steps=200;

1. Price European put options using both Black Scholes and implicit finite differences.

2. Price the American put option using implicit finite differences.

C++ program:

```cpp
double S = 50.0; double K = 50.0; double r = 0.1; double sigma = 0.4; double time=0.5;
int no_S_steps=200; int no_t_steps=200;
cout << " black scholes put price = " << option_price_put_black_scholes(S,K,r,sigma,time)<< endl;
cout << " implicit Euro put price = ";
cout << option_price_put_european_finite_diff_implicit(S,K,r,sigma,time,no_S_steps,no_t_steps) << endl;
cout << " implicit American put price = ";
cout << option_price_put_american_finite_diff_implicit(S,K,r,sigma,time,no_S_steps,no_t_steps) << endl;
```

Output from C++ program:

black scholes put price = 4.35166
implicit Euro put price = 4.34731
implicit American put price = 4.60064

Matlab program:

```matlab
S = 50.0;
K = 50.0;
r = 0.1;
sigma = 0.4;
time=0.5;
no_S_steps=200;
no_t_steps=200;
P= findiff_imp_am_put(S,K,r,sigma,time,no_S_steps,no_t_steps)
```

Output from Matlab program:

$P = 4.6006$

Exercise 13.2.

The Newmat library is only one of a large number of available matrix libraries, both public domain and commercial offerings. Look into alternative libraries and replace Newmat with one of these alternative libraries. What needs changing in the option price formulas?

13.9 References

Hull (2011). See the natbib documentation.
Chapter 14

Option pricing by simulation

Contents

14.1 Simulating lognormally distributed random variables .................................................. 143
14.2 Pricing of European Call options ................................................................................. 143
14.3 Hedge parameters ......................................................................................................... 144
14.4 More general payoffs. Function prototypes ................................................................. 146
14.5 Improving the efficiency in simulation ....................................................................... 147
  14.5.1 Control variates ....................................................................................................... 147
  14.5.2 Antithetic variates .................................................................................................. 148
14.6 More exotic options ...................................................................................................... 151
14.7 References .................................................................................................................. 152

We now consider using Monte Carlo methods to estimate the price of an European option, and let us first consider the case of the “usual” European Call, which can priced by the Black Scholes equation. Since there is already a closed form solution for this case, it is not really necessary to use simulations, but we use the case of the standard call for illustrative purposes.

At maturity, a call option is worth

\[ c_T = \max(0, S_T - X) \]

At an earlier date \( t \), the option value will be the expected present value of this.

\[ c_t = E[ PV(\max(0, S_T - X)) ] \]

Now, an important simplifying feature of option pricing is the “risk neutral result,” which implies that we can treat the (suitably transformed) problem as the decision of a risk neutral decision maker, if we also modify the expected return of the underlying asset such that this earns the risk free rate.

\[ c_t = e^{-r(T-t)} E^*[ \max(0, S_T - X) ] \]

where \( E^*[\cdot] \) is a transformation of the original expectation. One way to estimate the value of the call is to simulate a large number of sample values of \( S_T \) according to the assumed price process, and find the estimated call price as the average of the simulated values. By appealing to a law of large numbers, this average will converge to the actual call value, where the rate of convergence will depend on how many simulations we perform.
14.1 Simulating lognormally distributed random variables

Lognormal variables are simulated as follows. Let $\tilde{x}$ be normally distributed with mean zero and variance one. If $S_t$ follows a lognormal distribution, then the one-period-later price $S_{t+1}$ is simulated as

$$S_{t+1} = S_t e^{(r - \frac{1}{2} \sigma^2) + \sigma \tilde{x}},$$

or more generally, if the current time is $t$ and terminal date is $T$, with a time between $t$ and $T$ of $(T-t)$,

$$S_T = S_t e^{(r - \frac{1}{2} \sigma^2)(T-t) + \sigma \sqrt{T-t} \tilde{x}}.$$

Simulation of lognormal random variables is illustrated by C++ Code 14.1.

```cpp
#include <cmath>
using namespace std;

#include "normdist.h"

double simulate_lognormal_random_variable(const double& S, // current value of variable
                                          const double& r,  // interest rate
                                          const double& sigma, // volatility
                                          const double& time) { // time to final date

    double R = (r - 0.5 * pow(sigma, 2)) * time;
    double SD = sigma * sqrt(time);
    return S * exp(R + SD * random_normal());
}
```

C++ Code 14.1: Simulating a lognormally distributed random variable

14.2 Pricing of European Call options

For the purposes of doing the Monte Carlo estimation of the price of an European call

$$c_t = e^{-r(T-t)} E[\max(0, S_T - X)],$$

note that here one merely need to simulate the terminal price of the underlying, $S_T$, the price of the underlying at any time between $t$ and $T$ is not relevant for pricing. We proceed by simulating lognormally distributed random variables, which gives us a set of observations of the terminal price $S_T$. If we let $S_{T,1}, S_{T,2}, S_{T,3}, \ldots, S_{T,n}$ denote the $n$ simulated values, we will estimate $E^*[\max(0, S_T - X)]$ as the average of option payoffs at maturity, discounted at the risk free rate.

$$\hat{c}_t = e^{-r(T-t)} \left( \sum_{i=1}^{n} \max(0, S_{T,i} - X) \right)$$

C++ Code 14.2 shows the implementation of a Monte Carlo estimation of an European call option.

Example

Given $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$, time=$1$. Use 5000 simulations. Price put and call option using Black Scholes and simulation.

143
14.3 Hedge parameters

It is of course, just as in the standard case, desirable to estimate hedge parameters as well as option prices. We will show how one can find an estimate of the option delta, the first derivative of the call price with respect to the underlying security: \( \Delta = \frac{\partial c}{\partial S} \). To understand how one goes about estimating this, let us recall that the first derivative of a function \( f \) is defined as the limit

\[
 f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

Thinking of \( f(S) \) as the option price formula \( c_t = f(S; X, r, \sigma, (T - t)) \), we see that we can evaluate the option price at two different values of the underlying, \( S \) and \( S + q \), where \( q \) is a small quantity, and estimate the option delta as

\[
 \tilde{\Delta} = \frac{f(S + q) - f(S)}{q}
\]
In the case of Monte Carlo estimation, it is very important that this is done by using the same sequence of random variables to estimate the two option prices with prices of the underlying $S$ and $S + q$. C++ Code 14.3 implements this estimation of the option delta.

```
#include <cmath> // standard mathematical functions
#include <algorithm> // define the max() function
using namespace std;
#include "normdist.h" // definition of random number generator

double option_price_delta_call_european_simulated(const double& S, const double& K, const double& r, const double& sigma, const double& time, const int& no_sims){
  double R = (r - 0.5 * pow(sigma,2))*time;
  double SD = sigma * sqrt(time);
  double sum_payoffs = 0.0;
  double sum_payoffs_q = 0.0;
  double q = S*0.01;
  for (int n=1; n<=no_sims; n++) {
    double Z = random_normal();
    double S_T = S*exp(R + SD * Z);
    sum_payoffs += max(0.0, S_T-K);
    double S_T_q = (S+q)*exp(R + SD * Z);
    sum_payoffs_q += max(0.0, S_T_q-K);
  }
  double c = exp(-r*time) * (sum_payoffs/no_sims);
  double c_q = exp(-r*time) * (sum_payoffs_q/no_sims);
  return (c_q-c)/q;
}
```

C++ Code 14.3: Estimate Delta of European Call option priced by Monte Carlo

One can estimate other hedge parameters in a similar way.

Example

Given $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$, time=1. Use 5000 simulations. Calculate deltas of put and call option using Black Scholes and simulation.

```
C++ program:

double S=100.0; double K=100.0; double r=0.1; double sigma=0.25;
double time=1.0; int no_sims=5000;
cout << " call: bs delta = " << option_price_delta_call_black_scholes(S,K,r,sigma,time)
  << " sim delta = " << option_price_delta_call_european_simulated(S,K,r,sigma,time,no_sims)
  << endl;
cout << " put: bs delta = " << option_price_delta_put_black_scholes(S,K,r,sigma,time)
  << " sim delta = " << option_price_delta_put_european_simulated(S,K,r,sigma,time,no_sims)
  << endl;
```

Output from C++ program:

call: bs delta = 0.700208 sim delta = 0.701484
put: bs delta = -0.299792 sim delta = -0.307211
14.4 More general payoffs. Function prototypes

The above shows the case for a call option. If we want to price other types of options, with different payoffs we could write similar routines for every possible case. But this would be wasteful, instead a bit of thought allows us to write option valuations for any kind of option whose payoff depend on the value of the underlying at maturity, only. Let us now move toward a generic routine for pricing derivatives with Monte Carlo. This relies on the ability of C++ to write subroutines which one call with function prototypes, i.e. that in the call to to the subroutine/function one provides a function instead of a variable. Consider pricing of standard European put and call options. At maturity each option only depend on the value of the underlying \( S_T \) and the exercise price \( X \) through the relations

\[
C_T = \max(S_T - X, 0)
\]

\[
P_T = \max(X - S_T, 0)
\]

C++ Code 14.4 shows two C++ functions which calculates this.

```cpp
#include <algorithm>
using namespace std;

double payoff_call(const double & S, const double & K) {
    return max(0.0, S - K);
}

double payoff_put (const double & S, const double & K) {
    return max(0.0, K - S);
}
```

C++ Code 14.4: Payoff call and put options

The interesting part comes when one realises one can write a generic simulation routine to which one provide one of these functions, or some other function describing a payoff which only depends on the price of the underlying and some constant. C++ Code 14.5 shows how this is done.

```cpp
#include <cmath>
using namespace std;
#include "fin_recipes.h"

double derivative_price_simulate_european_option_generic(const double & S,
    const double & K,
    const double & r,
    const double & sigma,
    const double & time,
    double payoff(const double & price, const double & X),
    const int & no_sims) {

double sum_payoffs=0;
for (int n=0; n<no_sims; n++) {
    double S_T = simulate_lognormal_random_variable(S, r, sigma, time);
    sum_payoffs += payoff(S_T, K);
}
return exp(-r*time) * (sum_payoffs/no_sims);
}
```

C++ Code 14.5: Generic simulation pricing

Note the presence of the line
double payoff(const double& S, const double& K),

in the subroutine call. When this function is called, the calling program will need to provide a function
to put there, such as the Black Scholes example above. The next example shows a complete example of
how this is done.

Example
Given $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$, time=1, no sims=5000.

C++ program:

```cpp
double S = 100; double K = 100; double r = 0.1;
double sigma = 0.25; double time = 1.0; int no_sims = 50000;
cout << "Black Scholes call option price = " << option_price_call_black_scholes(S,K,r,sigma,time) << endl;
cout << "Simulated call option price = "
    << derivative_price_simulate_european_option_generic(S,K,r,sigma,time,payoff_call,no_sims)
    << endl;
cout << "Black Scholes put option price = " << option_price_put_black_scholes(S,K,r,sigma,time) << endl;
cout << "Simulated put option price = "
    << derivative_price_simulate_european_option_generic(S,K,r,sigma,time,payoff_put,no_sims)
    << endl;
```

Output from C++ program:

- Black Scholes call option price = 14.9758
- Simulated call option price = 14.995
- Black Scholes put option price = 5.45954
- Simulated put option price = 5.5599

As we see, even with as many as 50,000 simulations, the option prices estimated using Monte Carlo still
differs substantially from the “true” values.

### 14.5 Improving the efficiency in simulation

There are a number of ways of “improving” the implementation of Monte Carlo estimation such that the
estimate is closer to the true value.

#### 14.5.1 Control variates.

One is the method of control variates. The idea is simple. When one generates the set of terminal values
of the underlying security, one can value several derivatives using the same set of terminal values. What
if one of the derivatives we value using the terminal values is one which we have an analytical solution
to? For example, suppose we calculate the value of an at the money European call option using both
the (analytical) Black Scholes formula and Monte Carlo simulation. If it turns out that the Monte Carlo
estimate overvalues the option price, we think that this will also be the case for other derivatives valued
using the same set of simulated terminal values. We therefore move the estimate of the price of the
derivative of interest downwards.

Thus, suppose we want to value an European put and we use the price of an at the money European
call as the control variate. Using the same set of simulated terminal values $S_{T,i}$, we estimate the two
options using Monte Carlo as:

$$
\hat{p}_i = e^{-r(T-t)} \left( \sum_{i=1}^{n} \max (0, X - S_{T,i}) \right)
$$

147
We calculate the Black Scholes value of the call $c_t$, and calculate $p_t^{cv}$, the estimate of the put price with a control variate adjustment, as follows

$$p_t^{cv} = p_t + \left( c_t^{bs} - c_t \right)$$

One can use other derivatives than the at-the-money call as the control variate, the only limitation being that it has a tractable analytical solution.

**C++ Code 14.6** shows the implementation of a Monte Carlo estimation using an at-the-money European call as the control variate.

```cpp
#include <cmath>
#include "fin_recipes.h"

double
derivative_price_simulate_european_option_generic_with_control_variate(const double& S,
                              const double& X,
                              const double& r,
                              const double& sigma,
                              const double& time,
                              double payoff(const double& S,
                                             const double& X),
                              const int& no_sims) {

double c_bs = option_price_call_black_scholes(S,S,sigma,time); // price an at the money Black Scholes call
double sum_payoffs=0;
double sum_payoffs_bs=0;
for (int n=0; n<no_sims; n++) {
    double S_T= simulate_lognormal_random_variable(S,r,sigma,time);
    sum_payoffs += payoff(S_T,X);
    sum_payoffs_bs += payoff_call(S_T,S); // simulate at the money Black Scholes price
};
double c_sim = exp(-r*time) * (sum_payoffs/no_sims);
double c_bs_sim = exp(-r*time) * (sum_payoffs_bs/no_sims);
c_sim += (c_bs-c_bs_sim);
return c_sim;
};
```

**C++ Code 14.6**: Generic with control variate

### 14.5.2 Antithetic variates.

An alternative to using control variates is to consider the method of *antithetic* variates. The idea behind this is that Monte Carlo works best if the simulated variables are “spread” out as closely as possible to the true distribution. Here we are simulating unit normal random variables. One property of the normal is that it is symmetric around zero, and the median value is zero. Why don’t we enforce this in the simulated terminal values? An easy way to do this is to first simulate a unit normal random variable $Z$, and then use both $Z$ and $-Z$ to generate the lognormal random variables. **C++ Code 14.7** shows the implementation of this idea.

Boyle (1977) shows that the efficiency gain with antithetic variates is not particularly large. There are other ways of ensuring that the simulated values really span the whole sample space, sometimes called “pseudo Monte Carlo.”
#include "fin_recipes.h"
#include "normdist.h"
#include <cmath>
using namespace std;

double derivative_price_simulate_european_option_generic_with_antithetic_variate(const double& S,
    const double& K,
    const double& r,
    const double& sigma,
    const double& time,
    double payoff(const double& S, const double& X),
    const int& no_sims)
    {
        double R = (r - 0.5 * pow(sigma,2)) * time;
        double SD = sigma * sqrt(time);
        double sum_payoffs = 0;
        for (int n = 0; n < no_sims; n++)
        {
            double x = random_normal();
            double S1 = S * exp(R + SD * x);
            sum_payoffs += payoff(S1, K);
            double S2 = S * exp(R + SD * (-x));
            sum_payoffs += payoff(S2, K);
        }
        return exp(-r * time) * (sum_payoffs / (2 * no_sims));
    }

C++ Code 14.7: Generic with antithetic variates
Example

Given $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$, time = 1, no sims = 50000. Price put and call option using Black Scholes and simulation. The simulation is to use both control variates and antithetic methods.

C++ program:

```cpp
double S = 100; double K = 100; double r = 0.1;
double sigma = 0.25; double time = 1; int no_sims = 50000;

cout << "Black Scholes call option price = " << option_price_call_black_scholes(S,K,r,sigma,time) << endl;
cout << "Simulated call option price = " << derivative_price_simulate_european_option_generic(S,K,r,sigma,time, payoff_call,no_sims) << endl;
cout << "Simulated call option price, CV = " << derivative_price_simulate_european_option_generic_with_control_variate(S,K,r,sigma,time, payoff_call,no_sims) << endl;
cout << "Simulated call option price, AV = " << derivative_price_simulate_european_option_generic_with_antithetic_variate(S,K,r,sigma,time, payoff_call,no_sims) << endl;

cout << "Black Scholes put option price = " << option_price_put_black_scholes(S,K,r,sigma,time) << endl;
cout << "Simulated put option price = " << derivative_price_simulate_european_option_generic(S,K,r,sigma,time,payoff_put,no_sims) << endl;
cout << "Simulated put option price, CV = " << derivative_price_simulate_european_option_generic_with_control_variate(S,K,r,sigma,time, payoff_put,no_sims) << endl;
cout << "Simulated put option price, AV = " << derivative_price_simulate_european_option_generic_with_antithetic_variate(S,K,r,sigma,time, payoff_put,no_sims) << endl;
```

Output from C++ program:

Black Scholes call option price = 14.9758
Simulated call option price = 14.995
Simulated call option price, CV = 14.9758
Simulated call option price, AV = 14.9919
Black Scholes put option price = 5.45954
Simulated put option price = 5.41861
Simulated put option price, CV = 5.42541
Simulated put option price, AV = 5.46043
14.6 More exotic options

These generic routines can also be used to price other options. Any European option that only depends on the terminal value of the price of the underlying security can be valued. Consider the binary options discussed by e.g. Hull (2011). An cash or nothing call pays a fixed amount $Q$ if the price of the asset is above the exercise price at maturity, otherwise nothing. An asset or nothing call pays the price of the asset if the price is above the exercise price at maturity, otherwise nothing. Both of these options are easy to implement using the generic routines above, all that is necessary is to provide the payoff functions as shown in C++ Code 14.8.

```cpp
double payoff_cash_or_nothing_call(const double& S, const double& K){
    if (S>=K) return 1;
    return 0;
};

double payoff_asset_or_nothing_call(const double& S, const double& K){
    if (S>=K) return S;
    return 0;
};
```

C++ Code 14.8: Payoff binary options

Now, many exotic options are not simply functions of the terminal price of the underlying security, but depend on the evolution of the price from "now" till the terminal date of the option. For example options that depend on the average of the price of the underlying (Asian options). For such cases one will have to simulate the whole path. We will return to these cases in the chapter on pricing of exotic options.

Example

Given $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$, time=1. Use 5000 simulations. Price cash or nothing option with $Q = 1$.

Price asset or nothing option.

In both cases compare results using control variate and antithetic methods.
Exercise 14.1.

Consider the pricing of an European Call option as implemented in code 14.2, and the generic formula for pricing with Monte Carlo for European options that only depend on the terminal value of the underlying security, as implemented in code 14.5. Note the difference in the implementation of the lognormal simulation of terminal values. Why can one argue that the first implementation is more efficient than the other?

14.7 References

Boyle (1977) is a good early source on the use of the Monte Carlo technique for pricing derivatives. Simulation is also covered in Hull (2011).
Chapter 15

Pricing American Options – Approximations

There has been developed some useful approximations to various specific options. It is of course American options that are approximated. We will look at a number of approximations, both for their intrinsic usefulness, but also as examples of different approaches to generating an approximated value. The first we will look at is the Johnson (1983) approximation to an American put. This is a purely empirical approach to estimating the functional forms, with parameters estimated using regressions. The Geske and Johnson (1984) approach is to view the American option as the limit of a sequence of Bermudan options with increasing number of exercise dates. The American value is found by interpolating forward the Bermudan values. The Barone-Adesi and Whaley (1987) approximation decomposes the American option into two, the European value, and the early exercise premium. The early exercise premium is relatively small, and is easier to approximate. Finally, the Bjerksund and Stensland (1993) is an example of bounding the option value by finding a lower bound on its value. A typical approach to doing so is to specify some (suboptimal) exercise strategy. As soon as the exercise strategy is known, the option is easily valued.

Approximations are useful, but they should be used with caution. Typically, an approximation works well within a range of parameter values, but can go seriously wrong for some unfortunate combination of parameters. The user is well advised to to consider alternative approximations, and at a minimum use the price of the corresponding European option as a “sanity check” in the calculations. (In fact, in several of the following routines values are checked against Black Scholes values.)

15.1 The Johnson (1983) approximation

An early attempt at an analytical approximation to the price of an American put is provided by Johnson (1983). Formula 15.1 provides the details, and C++ Code 15.1 provides the implementation.

Example

Using the parameters $r = 0.125$, $S = 1.1$, $X = 1$, $\sigma = 0.5$ and time to maturity of one year, calculate the price of an american call using the Johnson (1983) approximation.
The constants \( a_0, a_1, b_1 \) and \( b_2 \) are constants. Their parameter values are estimated as

\[
\begin{align*}
a_0 &= 3.9649 & a_1 &= 0.032325 \\
b_0 &= 1.040803 & b_1 &= 0.00963
\end{align*}
\]

\( S_c \): Current value of underlying security.
\( X \): Exercise price of American put.
\( \sigma \): Volatility of underlying.
\( r \): Risk free interest rate.
\( (T - t) \): Time to maturity for option.

Formula 15.1: The functional form of the Johnson (1983) approximation to the value of an American put

\[ P(X) = \alpha p\left( X e^{r(T-t)} \right) + (1 - \alpha)p(X) \]

\[ \alpha = \left( \frac{r(T-t)}{a_0 r(T-t) + a_1} \right)^t, \text{ where } t = \frac{\ln(S/S_c)}{\ln(X/S_c)} \]

\[ S_c = X \left( \frac{\gamma}{1 + \gamma} \right)^m, \text{ where } m = \frac{\sigma^2(T-t)}{b_0 \sigma^2(T-t) + b_1} \]

\( \gamma = 2r/\sigma^2 \)

Barone-Adesi (2005) notes that this approximation is inaccurate outside of the parameter range considered in the paper.
#include <cmath>
#include "fin_recipes.h"

double option_price_american_put_approximated_johnson( const double & S, 
              const double & X, 
              const double & r, 
              const double & sigma, 
              const double & time ){

double sigma_sqr=pow(sigma,2);
double a0= 3.9649 ;
double a1 = 0.032325;
double b0 = 1.040803;
double b1 = 0.00963;
double gamma = 2*r/sigma_sqr;
double m = (sigma_sqr*time)/(b0*sigma_sqr*time+b1);
double Sc = X * pow ((gamma)/(1+gamma)).m;
double l = (log(S/Sc))/(log(X/Sc) );
double alpha = pow( ( t*time)/(a0*t*time+a1) ), l );
double P = alpha*option_price_put_black_scholes(S,X*exp(r*time),r,sigma, time) 
        + (1-alpha)*option_price_put_black_scholes(S,X,r,sigma, time); // for safety use the Black Scholes as lower bound

double p=option_price_put_black_scholes(S,X,r,sigma, time); // for safety use the Black Scholes as lower bound

return max(p,P);
};

C++ Code 15.1: The Johnson (1983) approximation to an american put price
15.2 An approximation to the American Put due to Geske and Johnson (1984)

Geske and Johnson (1984) develop an approximation for the American put problem. The solution technique is to view the American put as an sequence of Bermudan options, with the number of exercise dates increasing. The correct value is the limit of this sequence.

Define \( P_i \) to be the price of the put option with \( i \) dates of exercise left. \( P_1 \) is then the price of an European option, with the one exercise date the expiry date. \( P_2 \) is the price of an option that can be exercised twice, once halfway between now and expiry, the other at expiry. Geske-Johnson shows how these options may be priced, and then develops a sequence of approximate prices that converges to the correct price. An approximation involving 3 option evaluations is

\[
P = P_3 + \frac{7}{2}(P_3 - P_2) - \frac{1}{2}(P_2 - P_1)
\]

To calculate these, first define

\[
d_1(q, \tau) = \frac{\ln \left( \frac{S}{q} \right) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma \sqrt{\tau}}, \quad d_2(q, \tau) = d_1 - \sigma \sqrt{\tau}
\]

and

\[
\rho_{12} = \frac{1}{\sqrt{2}}, \quad \rho_{13} = \frac{1}{\sqrt{3}}, \quad \rho_{23} = \sqrt{\frac{2}{3}}
\]

In this notation, \( P_1 \) is the ordinary (European) Black Scholes value:

\[
P_1 = X e^{-rT} N_1(-d_2(X, T)) - SN_1(d_1(X, T))
\]

To calculate \( P_2 \):

\[
P_2 = X e^{-r\frac{T}{2}} N_1 \left( -d_2 \left( \frac{S}{u}, \frac{T}{2} \right) \right) - SN_1 \left( -d_1 \left( \frac{S}{u}, \frac{T}{2} \right) \right) + X e^{-rT} N_2 \left( d_2 \left( \frac{S}{u}, \frac{T}{2} \right), -d_2(X, T); -\rho_{12} \right) - SN_2 \left( d_1 \left( \frac{S}{u}, \frac{T}{2} \right), -d_1(X, T); -\rho_{12} \right)
\]

and \( S \) solves

\[
S = X - p \left( S, X, \frac{T}{2}, \tau, \sigma \right) = \frac{S}{u}
\]

To calculate \( P_3 \):

\[
P_3 = X e^{-r\frac{T}{3}} N_1 \left( -d_2 \left( \frac{S}{w}, \frac{T}{3} \right) \right) - SN_1 \left( -d_1 \left( \frac{S}{w}, \frac{T}{3} \right) \right) + X e^{-r\frac{2T}{3}} N_2 \left( d_2 \left( \frac{S}{w}, \frac{T}{3} \right), -d_2 \left( \frac{S}{w}, \frac{2T}{3} \right); -\rho_{12} \right) - SN_2 \left( d_1 \left( \frac{S}{w}, \frac{T}{3} \right), -d_1 \left( \frac{S}{w}, \frac{2T}{3} \right); -\rho_{12} \right) + X e^{-rT} N_3 \left( d_1 \left( \frac{S}{u}, \frac{T}{3} \right), d_1 \left( \frac{S}{w}, \frac{2T}{3} \right), -d_1(X, T); \rho_{12}, -\rho_{13}, -\rho_{23} \right) - SN_3 \left( d_2 \left( \frac{S}{u}, \frac{T}{3} \right), d_2 \left( \frac{S}{w}, \frac{2T}{3} \right), -d_2(X, T); \rho_{12}, -\rho_{13}, -\rho_{23} \right)
\]
where the critical stock prices solves

\[ S = X - P_2 \left( S, X, \frac{2T}{3}, r, \sigma \right) = \mathbb{E}_{T/3} \]

\[ S = X - p \left( S, X, \frac{T}{3}, r, \sigma \right) = \mathbb{E}_{2T/3} \]

The main issue in implementation of this formula (besides keeping the notation straight) is the evaluation of \( P_3 \), since it involves the evaluation of a trivariate normal cumulative distribution. The evaluation of \( N_3() \) is described later, since it involves calls to external routines for numerical integration.

To solve for

\[ S = X - p \left( S, X, \frac{T}{2}, r, \sigma \right) = \mathbb{E}_T \]

involves Newton steps. Define

\[ g(S) = S - X + p \left( S, X, \frac{T}{2}, r, \sigma \right) \]

Iterate on \( S \)

\[ S_i = S_i - \frac{g}{g'} \]

until \( g \) is equal to zero with some given tolerance.
#include <cmath>
#include "normdist.h"
#include "fin_recipes.h"
#include <iostream>

const double ACCURACY=1e-6;

inline double d1(const double &S, const double &r, const double &sigma, const double &tau){
    return (log(S/X) + (r+0.5*pow(sigma,2)))*tau/(sigma*sqrt(tau));
}

inline double d2(const double &S, const double &X, const double &r, const double &sigma, const double &tau){
    return d1(S,X,r,sigma,tau) - sigma*sqrt(tau);
}

inline double calcP2(const double &S, const double &X, const double &r, const double &sigma, const double &time, const double &t2, const double &S2_bar, const double &rho12){
    double P2 = X*exp(-r*t2)*N(-d2(S,S2_bar,r,sigma,t2));
    P2 += X*exp(-r*time)*N(d2(S,S2_bar,r,sigma,t2), -d2(S,X,r,sigma,1.0),-rho12);
    P2 = *N(d1(S,S2_bar,r,sigma,t2), -d1(S,X,r,sigma,1.0),-rho12);
    return P2;
}

do}{double option_price_american_put_approximated_geske_johnson(const double &S, const double &X, const double &r, const double &sigma, const double &time)
{
    double P1 = option_price_put_black_scholes(S,X,r,sigma,1.0);
    double rho12=1.0/sqrt(2.0); double rho23=1.0/sqrt(3.0);
    double t2 = time/2.0; double t3 = time/3.0;
    double Si=S; double S2_bar=S;
    double g=1; double gprime=1;
    while(fabs(g)>ACCURACY)
    {
        g=Si-X+option_price_put_black_scholes(Si,X,r,sigma,t2);
        gprime=1+option_price_delta_put_black_scholes(Si,X,r,sigma,t2);
        S2_bar=Si;
        Si=Si-g/gprime;
    }
    double P2 = calcP2(S,X,r,sigma,time,t2,S2_bar,rho12);
    P2=max(P1,P2); // for safety, use one less step as lower bound
    double S23_bar=S2_bar;
    g=1;
    while(fabs(g)>ACCURACY)
    {
        g=Si-X+option_price_put_black_scholes(Si,X,r,sigma,t3);
        gprime=1+option_price_delta_put_black_scholes(Si,X,r,sigma,t3);
        S3_bar=Si;
        Si=Si-g/gprime;
    }
    double P3 = X * exp(-r*t3) * N(-d2(S,S3_bar,r,sigma,t3));
    P3 = S * N(-d1(S,S3_bar,r,sigma,t3));
    P3 += X*exp(-r*time)*N(d2(S,S3_bar,r,sigma,t3), -d2(S,S3_bar,r,sigma,t3), -rho12);
    P3 += X*exp(-r*time)*N(d2(S,S3_bar,r,sigma,t3), -d2(S,X,r,sigma,1.0), -rho12);
    P3 += X*exp(-r*time)*N3(d1(S,S3_bar,r,sigma,t3),d1(S,S3_bar,r,sigma,t3),-d1(S,X,r,sigma,1.0), rho12,-rho13,-rho23);
    P3 = *N3(d2(S,S3_bar,r,sigma,t3),d2(S,S3_bar,r,sigma,t3),-d2(S,X,r,sigma,1.0),rho12,-rho13,-rho23);
    P3=max(P2,P3); // for safety, use one less step as lower bound
    return P3+3.5*(P3-P2)-0.5*(P2-P1);
}

C++ Code 15.2: Geske Johnson approximation of American put
15.3 A quadratic approximation to American prices due to Barone–Adesi and Whaley.

We now discuss an approximation to the option price of an American option on a commodity, described in Barone-Adesi and Whaley (1987) (BAW). The commodity is assumed to have a continuous payout $b$. The starting point for the approximation is the (Black-Scholes) stochastic differential equation valid for the value of any derivative with price $V$.

$$\frac{1}{2} \sigma^2 S^2 V_S S + b S V_S - r V + V_t = 0 \quad (15.1)$$

Here $V$ is the (unknown) formula that determines the price of the contingent claim. For an European option the value of $V$ has a known solution, the adjusted Black Scholes formula. For American options, which may be exercised early, there is no known analytical solution.

To do their approximation, BAW decomposes the American price into the European price and the early exercise premium

$$C(S, T) = c(S, T) + \epsilon_C(S, T)$$

Here $\epsilon_C$ is the early exercise premium. The insight used by BAW is that $\epsilon_C$ must also satisfy the same partial differential equation. To come up with an approximation BAW transformed equation (15.1) into one where the terms involving $V_t$ are negligible, removed these, and ended up with a standard linear homogeneous second order equation, which has a known solution.

The functional form of the approximation is shown in formula 15.2.

$$C(S, T) = \begin{cases} 
  c(S, T) + A_2 \left( \frac{S}{S^*} \right)^{q_2} & \text{if } S < S^* \\
  S - X & \text{if } S \geq S^*
\end{cases}$$

where

$$q_2 = \frac{1}{2} \left( -(N - 1) + \sqrt{(N - 1)^2 + \frac{4M}{K}} \right)$$

$$A_2 = \frac{S^*}{q_2} \left( 1 - e^{(b-r)(T-t)} N (d_1(S^*)) \right)$$

$$M = \frac{2r}{q_2}, \quad N = \frac{2b}{q_2}, \quad K(T) = 1 - e^{-r(T-t)}$$

and $S^*$ solves

$$S^* - X = c(S^*, T) + \frac{S^*}{q_2} \left( 1 - e^{(b-r)(T-t)} N (d_1(S^*)) \right)$$

Notation: $S$ Stock price. $X$: Exercise price.

Formula 15.2: The functional form of the Barone Adesi Whaley approximation to the value of an American call

In implementing this formula, the only problem is finding the critical value $S^*$. This is the classical problem of finding a root of the equation

$$g(S^*) = S^* - X - c(S^*) - \frac{S^*}{q_2} \left( 1 - e^{(b-r)(T-t)} N (d_1(S^*)) \right) = 0$$

The approximation is also discussed in Hull (2011).
This is solved using Newton’s algorithm for finding the root. We start by finding a first “seed” value $S_0$. The next estimate of $S_i$ is found by:

$$S_{i+1} = S_i - \frac{g(S_i)}{g'(S_i)}$$

At each step we need to evaluate $g(S)$ and its derivative $g'(S)$.

$$g(S) = S - X - c(S) - \frac{1}{\sqrt{2\pi}} S \left(1 - e^{(b-r)(T-t)} N(d_1)\right)$$

$$g'(S) = (1 - \frac{1}{\sqrt{2\pi}} \left(1 - e^{(b-r)(T-t)} N(d_1)\right)) + \frac{1}{\sqrt{2\pi}} (e^{(b-r)(T-t)} n(d_1)) \frac{1}{\sigma \sqrt{T-t}}$$

where $c(S)$ is the Black Scholes value for commodities. Code 15.3 shows the implementation of this formula for the price of a call option.

**Example**

Consider the following set of parameters, used as an example in the Barone-Adesi and Whaley (1987) paper: $S = 100$, $X = 100$, $\sigma = 0.20$, $r = 0.08$, $b = -0.04$.

1. Price a call option with time to maturity of 3 months.

   **C++ program:**

   ```cpp
   double S = 100; double X = 100; double sigma = 0.20;
   double r = 0.08; double b = -0.04; double time = 0.25;
   cout << "Call price using Barone-Adesi Whaley approximation = "
   \qquad << option_price_american_call_approximated_baw(S,X,r,sigma,time) << endl;
   ```

   **Output from C++ program:**

   Call price using Barone-Adesi Whaley approximation = 5.74339

**Exercise 15.1.**

The Barone-Adesi – Whaley insight can also be used to value a put option, by approximating the value of the early exercise premium. For a put option the approximation is

$$P(S) = \frac{p(S,T) + A_1 \left(\frac{S}{S^*}\right)^{\gamma}}{X - S} \quad \text{if} \quad S > S^*$$

$$A_1 = -\frac{S^*}{\sqrt{2\pi}} (1 - e^{(b-r)(T-t)} N(-d_1(S^*)))$$

One again solves iteratively for $S^*$, for example by Newton’s procedure, where now one would use

$$g(S) = X - S - p(S) + \frac{S}{\sqrt{2\pi}} \left(1 - e^{(b-r)(T-t)} N(-d_1)\right)$$

$$g'(S) = \left(\frac{1}{\sqrt{2\pi}} - 1\right) \left(1 - e^{(b-r)(T-t)} N(-d_1)\right) + \frac{1}{\sqrt{2\pi}} e^{(b-r)(T-t)} \frac{1}{\sigma \sqrt{T-t}} n(-d_1)$$

1. Implement the calculation of the price of an American put option using the BAW approach.

160
#include <cmath>
#include <algorithm>
using namespace std;
#include "normdist.h" // normal distribution
#include "fin_recipes.h" // define other option pricing formulas

const double ACCURACY = 1.0e-6;

double option_price_american_call_approximated_base(const double & S, const double & X, const double & r, const double & b, const double & sigma, double & time) {
    double sigma_sqr = sigma*sigma;
    double time_sqr = sqrt(time);
    double nn = 2.0*b/sigma_sqr;
    double m = 2.0*r/sigma_sqr;
    double K = 1.0-exp(-r*time);
    double q2 = (-nn-1)+sqrt(pow(nn-1,2.0)+(4.0*m/K))*0.5;
    double q2_inf = 0.5 * (-nn-1) + sqrt(pow(nn-1,2.0)+4.0*m); // seed value from paper
    double S_star_inf = X / (1.0 - 1.0/q2_inf);
    double h2 = -(b*time+2.0*sigma*time_sqr)*(X/(S_star_inf-X));
    double S_seed = X + (S_star_inf-X)*(1.0-exp(h2));

    int no_iterations = 0; // iterate on S to find S_star, using Newton steps
    double S = S_seed;
    double g = 1;
    double gprime = 1.0;
    while ((fabs(g) > ACCURACY) & (fabs(gprime) > ACCURACY) // to avoid exploding Newton's
        & (no_iterations++<500)
        & (S>0.0)) {
        double c = option_price_american_call_payoff(S,X,r,b,sigma,time);
        double d1 = (log(S/X)+(b+0.5*sigma_sqr)*time)/(sigma*time_sqr);
        g = (1.0-1.0/q2)*Si-X-c+(1.0/q2)*Si*exp((b-r)*time)*N(d1);
        gprime = (1.0-1.0/q2)*(1.0-exp((b-r)*time)*N(d1))
            + (1.0/q2)*exp((b-r)*time)*n(d1)*(1.0/(sigma*time_sqr));
        Si = Si-g/gprime;
    }
    double S_star = 0;
    if (fabs(g) > ACCURACY) { S_star = S_seed; } // did not converge
    else { S_star = Si; }
    double C = 0;
    double C = option_price_american_call_payoff(S,X,r,b,sigma,time);
    if (S>=S_star) {
        C = C-S-X;
    } else {
        double d1 = (log(S_star/X)+(b+0.5*sigma_sqr)*time)/(sigma*time_sqr);
        double A2 = (1.0-exp((b-r)*time)*N(d1))* (S_star/q2);
        C = c + A2*pow((S/S_star),q2);
    }
    return max(C,c); // know value will never be less than BS value
}

C++ Code 15.3: Barone Adesi quadratic approximation to the price of a call option
15.4 An alternative approximation to american options due to Bjerksund and Stensland (1993)

Bjerksund and Stensland (1993) provides an alternative approximation to the price of American options. Their approximation is an example of calculating a lower bound for the option value. Their valuation relies on using an exercise strategy that is known, and although suboptimal, close enough to the (unknown) exercise strategy that the approximated value is

The corresponding put option can be estimated as

\[ P(S, X, T, r, b, \sigma) = C(X, S, T, r - b, -b, \sigma) \]
The call option is found as
\[ C(S, K, T, r, b, \sigma, X) = \alpha(X)S^\beta - \alpha(X)\varphi(S, T|\beta, X, X) + \varphi(S, T|1, X, X) \]
\[ - \varphi(S, T|1, K, X) - X\varphi(S, T|0, X, X) + X\varphi(S, T|0, K, X) \]

where
\[ \alpha(X) = (X - K)X^{-\beta} \]
\[ \beta = \left( \frac{1}{2} - \frac{b}{\sigma^2} \right) + \sqrt{\left( \frac{b}{\sigma^2} - \frac{1}{2} \right)^2 + 2 \frac{r}{\sigma^2}} \]
\[ \varphi(S, T|\gamma, H, X) = e^{\gamma S^2} \left[ N(d_1) - \left( \frac{X}{S} \right)^{\kappa} N(d_2) \right] \]
\[ \lambda = \left( -r + \gamma b + \frac{1}{2} \gamma (\gamma - 1) \sigma^2 \right) T \]
\[ d_1 = -\frac{\ln(S/H) + (b + (\gamma - \frac{1}{2})\sigma^2) T}{\sigma \sqrt{T}} \]
\[ d_2 = -\frac{\ln(X^2/S H) + (b + (\gamma - \frac{1}{2})\sigma^2) T}{\sigma \sqrt{T}} \]
\[ \kappa = \frac{2b}{\sigma^2} + 2\gamma - 1 \]
\[ X_T = B_0 + \left( B_{10} - B_0 \right) \left( 1 - e^{h(T)} \right) \]
\[ h(T) = -\left( bT + 2 \sigma \sqrt{T} \right) \left( \frac{B_0}{B_{10} - B_0} \right) \]
or (suggested in a later paper Bjerkund and Stensland (2002))
\[ h(T) = -\left( bT + 2 \sigma \sqrt{T} \right) \left( \frac{K^2}{B_{10} - B_0} \right) \]
\[ B_{10} = \frac{\beta}{\beta - 1} K \]
\[ B_0 = \max \left( K, \left( \frac{r}{r - b} \right) K \right) \]

\[ S \]: Price of underlying security. \[ K \]: Exercise price.

**Formula 15.3**: The Bjerkund and Stensland (1993) lower bound approximation to the value of an American Call
#include "fin_recipes.h"
#include <cmath>
#include "normdist.h"

inline double phi(double S, double T, double gamma, double H, double X, double r, double b, double sigma)
{
    double sigma_sqr = pow(sigma, 2);
    double kappa = 2.0*b/sigma_sqr + 2.0*gamma - 1.0;
    double lambda = (-r + gamma * b + 0.5*gamma*(gamma-1.0)*sigma_sqr)*T; // check this, says lambda in text
    double d1 = -(log(S/H)+(b+(gamma-0.5)*sigma_sqr)*T)/(sigma*sqrt(T));
    double d2 = -(log((X*T)/(S*H))+(b+(gamma-0.5)*sigma_sqr)*T)/(sigma*sqrt(T));
    double phi = exp(lambda) * pow(S,gamma) * (N(d1) - pow((X/S),kappa) * N(d2));
    return phi;
}

double option_pricemerican_call_approximated_bjerksund_stensland(const double& S, const double& K, const double& r, const double& b, const double& sigma, const double& T)
{
    double sigma_sqr = pow(sigma, 2);
    double B0 = max(K, (r/(r-b))*K);
    double beta = (0.5 - b/sigma_sqr) + sqrt(pow((b/sigma_sqr-0.5), 2) + 2.0*r/sigma_sqr);
    double Binf = beta/(beta-1.0)*K;
    double hT = -(b*T + 2.0*sigma*sqrt(T))*((K*K)/(Binf-B0));
    double XT = B0+(Binf-B0)*(1.0-exp(hT));
    double alpha = (XT-K)*pow(XT,-beta);
    double C = alpha*phi(S,T,beta,XT,XT,b,r,b,sigma);
    C += phi(S,T,1,XT,XT,b,r,b,sigma);
    C -= phi(S,T,1,K,XT,XT,b,sigma);
    C -= K*phi(S,T,0,XT,XT,r,b,sigma);
    C += K*phi(S,T,0,K,XT,XT,r,b,sigma);
    double c = option_priceeuropean_call_payout(S,K,r,b,sigma,T); // for safety use the Black Scholes as lower bound
    return max(c, C);
}

C++ Code 15.4: Approximation of American Call due to Bjerksund and Stensland (1993)

#include "fin_recipes.h"

double option_priceamerican_put_approximated_bjerksund_stensland(const double& S, const double& X, const double& r, const double& q, const double& sigma, const double& T)
{
    return option_priceamerican_call_approximated_bjerksund_stensland(X, S, r-(-q), r-(-q), sigma, T);
}

C++ Code 15.5: Approximation of American put due to Bjerksund and Stensland (1993)
Exercise 15.2.

An option is “At the Money Forward” if the forward price $F$ equals the exercise price.

1. Show that a reasonable approximation to the Black Scholes value of a call option on a non-dividend paying asset that is “At the Money Forward” is

$$C = 0.4\sigma \sqrt{T - t} PV(F)$$

where $F$ is the forward price, $PV(\cdot)$ signifies the present value operator, $T - t$ is the time to maturity and $\sigma$ is the volatility.

15.5 Readings

Chapter 16

Average, lookback and other exotic options

Contents

16.1 Bermudan options .............................................. 166
16.2 Asian options ................................................. 169
16.3 Lookback options .............................................. 170
16.4 Monte Carlo Pricing of options whose payoff depend on the whole price path ....... 172
   16.4.1 Generating a series of lognormally distributed variables ...................... 172
16.5 Control variate .................................................. 175
16.6 References ...................................................... 176

We now look at a type of options that has received a lot of attention in later years. The distinguishing factor of these options is that they depend on the whole price path of the underlying security between today and the option maturity.

16.1 Bermudan options

A Bermudan option is, as the name implies, a mix of an European and American option. It is a standard put or call option which can only be exercised at discrete dates throughout the life of the option. The simplest way to do the pricing of this is again the binomial approximation, but now, instead of checking at every node whether it is optimal to exercise early, only check at the nodes corresponding to the potential exercise times. C++ Code 16.1 shows the calculation of the Bermudan price using binomial approximations. The times as which exercise can happen is passed as a vector argument to the routine, and in the binomial a list of which nodes exercise can happen is calculated and checked at every step.

Example

Price a Bermudan put. The following information is given \( S = 80, K = 100, r = 0.20; \) time = 1; \( \sigma = 0.25, \) steps = 500, \( q = 0.0, \) Potential exercise times = 0.25, 0.5 and 0.75.

\footnote{Since Bermuda is somewhere between America and Europe...}
C++ program:

```c++
double S=80; double K=100; double r = 0.20;
double time = 1.0; double sigma = 0.25;
int steps = 500;
double q=0.0;
vector<double> potential_exercise_times; potential_exercise_times.push_back(0.25);
potential_exercise_times.push_back(0.5); potential_exercise_times.push_back(0.75);
cout << " Bermudan put price = "
    << option_price_put_bermudan_binomial(S,K,r,q,sigma,time,potential_exercise_times,steps)
    << endl;
```

Output from C++ program:

Bermudan put price = 15.8869
#include <cmath>  // standard C mathematical library
#include <algorithm>  // defines the max() operator
#include <vector>  // STL vector templates
using namespace std;

double option_price_put_bermudan_binomial( const double & S,  
  const double & X,  
  const double & r,  
  const double & q,  
  const double & sigma,  
  const double & time,  
  const vector<double> & potential_exercise_times,  
  const int & steps) {

  double delta_t = time / steps;  
  double R = exp(r * delta_t);  
  double Rinv = 1.0 / R;  
  double u = exp(sigma * sqrt(delta_t));  
  double uu = u * u;  
  double d = 1.0 / u;  
  double p_up = (exp((r - q) * delta_t) - d) / (u - d);  
  double p_down = 1.0 - p_up;  
  vector<double> prices(steps + 1);  
  vector<double> put_values(steps + 1);  
  vector<int> potential_exercise_steps;  // create list of steps at which exercise may happen

  for (int i = 0; i < potential_exercise_times.size(); ++i) {  
    double t = potential_exercise_times[i];  
    if ( (t > 0.0) && (t < time) ) {  
      potential_exercise_steps.push_back(int(t / delta_t));  
    }  
  }

  prices[0] = S * pow(d, steps);  // fill in the endnodes.
  for (int i = 0; i < steps; ++i) prices[i] = uu * prices[i - 1];  
  for (int i = 0; i < steps; ++i) put_values[i] = max(0.0, (X - prices[i]));  // put payoffs at maturity

  for (int step = steps - 1; step > 0; --step) {  
    bool check_exercise_this_step = false;  
    for (int j = 0; j < potential_exercise_steps.size(); ++j) {  
      if (step == potential_exercise_steps[j]) { check_exercise_this_step = true;  
    }  
    for (int i = 0; i <= step; ++i) {  
      put_values[i] = (p_up * put_values[i + 1] + p_down * put_values[i]) * Rinv;  
      prices[i] = d * prices[i + 1];  
      if (check_exercise_this_step) put_values[i] = max(put_values[i], X - prices[i]);  
    }  
  }

  return put_values[0];
}

C++ Code 16.1: Binomial approximation to Bermudan put option
16.2 Asian options

The payoff depends on the average of the underlying price. An average price call has payoff

\[ C_T = \max(0, \overline{S} - X) \]

where \( \overline{S} \) is the average of the underlying in the period between \( t \) and \( T \).

Another Asian is the average strike call

\[ C_T = \max(0, S_T \overline{S}) \]

There are different types of Asians depending on how the average \( \overline{S} \) is calculated. For the case of \( S \) being lognormal and the average \( \overline{S} \) being a geometric average, there is an analytic formula due to Kemna and Vorst (1990). Hull (2011) also discusses this case. It turns out that one can calculate this option using the regular Black Scholes formula adjusting the volatility to \( \sigma / \sqrt{3} \) and the dividend yield to

\[ \frac{1}{2} \left( r + q + \frac{1}{6} \sigma^2 \right) \]

in the case of continuous sampling of the underlying price distribution.

C++ Code 16.2 shows the calculation of the analytical price of an Asian geometric average price call.

```cpp
#include <cmath>
using namespace std;
#include "normdist.h" // normal distribution definitions

double option_price_asian_geometric_average_price_call(const double & S,
                           const double & K,
                           const double & r,
                           const double & q,
                           const double & sigma,
                           const double & time){
    double sigma_sqr = pow(sigma,2);
    double adj_div_yield=0.5*(r+q+sigma_sqr/6.0);
    double adj_sigma_sqr = pow(adj_sigma,2);
    double time_sqrt = sqrt(time);
    double d1 = (log(S/K) + (r-adj_div_yield + 0.5*adj_sigma_sqr)*time)/(adj_sigma*time_sqrt);
    double d2 = d1 - (adj_sigma*time_sqrt);
    double call_price = S * exp(-adj_div_yield*time)* N(d1) - K * exp(-r*time) * N(d2);
    return call_price;
};
```

**C++ Code 16.2**: Analytical price of an Asian geometric average price call

**Example**

Relevant parameters: \( S = 100 \), \( K = 100 \), \( q = 0 \), \( r = 0.06 \), \( \sigma = 0.25 \) and time to maturity is one year. Price an Asian geometric average price call.

**C++ program**:

```cpp
double S=100; double K=100; double q=0;
double r=0.06; double sigma=0.25; double time=1.0;
cout << " Analytical geometric average = "
    << option_price_asian_geometric_average_price_call(S,K,r,q,sigma,time)
    << endl;
```

**Output from C++ program**:

Analytical geometric average = 6.74876
16.3 Lookback options

The payoff from lookback options depend on the maximum or minimum of the underlying achieved through the period. The payoff from the lookback call is the terminal price of the undelying less the minimum value

\[ C_T = \max(0, S_T - \min_{\tau \in [t,T]} S_\tau) \]

For this particular option an analytical solution has been found, due to Goldman, Sosin, and Gatto (1979), which is shown in Formula 16.1 and implemented in C++ Code 16.3.

\[
C = S e^{-q(T-t)} N(a_1) - S e^{-q(T-t)} \frac{\sigma^2}{2(r-q)} N(-a_1) - S_{min} e^{-r(T-t)} \left( N(a_2) - \frac{\sigma^2}{2(r-q)} e^{Y_1} N(-a_3) \right)
\]

\[
a_1 = \frac{\ln \left( \frac{S}{S_{min}} \right) + (r - q + \frac{1}{2} \sigma^2) (T - t)}{\sigma \sqrt{T - t}}
\]

\[
a_2 = a_1 - \sigma \sqrt{T - t}
\]

\[
a_3 = \frac{\ln \left( \frac{S}{S_{min}} \right) + (-r + q + \frac{1}{2} \sigma^2) (T - t)}{\sigma \sqrt{T - t}}
\]

\[
Y_1 = \frac{2 (r - q - \frac{1}{2} \sigma^2) \ln \left( \frac{S}{S_{min}} \right)}{\sigma^2}
\]

Formula 16.1: Analytical formula for a lookback call

```cpp
#include <cmath>
#include "normdist.h"

using namespace std;

double option_price_european_lookback_call(const double& S,
                                          const double& Smin,
                                          const double& q,
                                          const double& sigma,
                                          const double& time){
    if (r==q) return 0;
    double sigma_sqr = sigma*sigma;
    double time_sqrt = sqrt(time);
    double a1 = (log(S/Smin) + (r+q+sigma_sqr/2.0)*time)/(sigma*time_sqrt);
    double a2 = a1-sigma*time_sqrt;
    double a3 = (log(S/Smin) + (-r+q+sigma_sqr/2.0)*time)/(sigma*time_sqrt);
    double Y1 = 2.0 * (r-sigma_sqr/2.0)*log(S/Smin)/sigma_sqr;
    return S * exp(-q*time)*N(a1) - Smin * exp(-q*time)*exp(-q*Y1)*exp(Y1)*N(-a3);
}
```

C++ Code 16.3: Price of lookback call option

Example

Parameters \( S = 100, S_{min} = S = q = 0, r = 0.06, \sigma = 0.346, \) time = 1.0, Price an European lookback call.
### C++ program:

```cpp
double S=100; double Smin=S; double q = 0; double r = 0.06;
double sigma = 0.346; double time = 1.0;
cout << " Lookback call price = "
    << option_price_european_lookback_call(Smin,q,sigma,time) << endl;
```

### Output from C++ program:

```
Lookback call price = 27.0713
```
16.4 Monte Carlo Pricing of options whose payoff depend on the whole price path

Monte Carlo simulation can be used to price a lot of different options. The limitation is that the options should be European. American options can not be priced by simulation methods. There is (at least) two reasons for this. First, the optimal exercise policy would have to be known. But if the exercise policy was known there would be an analytical solution. Second, approximations using Monte Carlo relies on a law of large numbers. But if some of the sample paths were removed, the LLN would be invalidated. In chapter 14 we looked at a general simulation case where we wrote a generic routine which we passed a payoff function to, and the payoff function was all that was necessary to define an option value. The payoff function in that case was a function of the terminal price of the underlying security. The only difference to the previous case is that we now have to generate a price sequence and write the terminal payoff of the derivative in terms of that, instead of just generating the terminal value of the underlying security from the lognormal assumption.

16.4.1 Generating a series of lognormally distributed variables

Recall that one will generate lognormally distributed variables as

\[ S_T = S_t e^{(r - \frac{1}{2} \sigma^2)(T-t) + \sigma \sqrt{T-t} \xi} \]

where the current time is \( t \) and terminal date is \( T \). To simulate a price sequence one splits this period into say \( N \) periods, each of length

\[ \Delta t = \frac{T - t}{N} \]

\[ \begin{array}{cccccccc}
  t & t + \Delta t & t + 2\Delta t & t + 3\Delta t & \cdots & T \\
  \end{array} \]

Time

Each step in the simulated price sequence is

\[ S_{t+\Delta t} = S_t e^{(r - \frac{1}{2} \sigma^2)\Delta t + \sigma \sqrt{\Delta t} \xi} \]

C++ Code 16.4 shows how one would simulate a sequence of lognormally distributed variables. This code is then used in the generic routine to do calculations, as shown in C++ Code 16.5.

To price an option we are then only in need of a definition of a payoff function. We consider a couple of examples. One is the case of an Asian option, shown in C++ Code 16.6.\(^2\)

Another is the payoff for a lookback, shown in C++ Code 16.7.\(^3\)

\(^2\)Note the use of the accumulate() function, which is part of the C++ standard.

\(^3\)Note the use of the min_element() and max_element functions, which are part of the C++ standard.
#include <cmath>
#include <vector>
using namespace std;
#include "normdist.h"

vector<double> simulate_lognormally_distributed_sequence(const double& S,
    const double& r,
    const double& sigma,
    const double& time, // time to final date
    const int& no_steps){ // number of steps
    vector<double> prices(no_steps);
    double delta_t = time/no_steps;
    double R = (r-0.5*pow(sigma,2))*delta_t;
    double SD = sigma * sqrt(delta_t);
    double S_t = S; // initialize at current price
    for (int i=0; i<no_steps; ++i) {
        S_t = S_t * exp(R + SD * random_normal());
        prices[i]=S_t;
    }
    return prices;
}

---

vector<double> prices(no_steps);
double delta_t = time/no_steps;
double R = (r-0.5*pow(sigma,2))*delta_t;
double SD = sigma * sqrt(delta_t);
double S_t = S; // initialize at current price
for (int i=0; i<no_steps; ++i) {
    S_t = S_t * exp(R + SD * random_normal());
    prices[i]=S_t;
}
return prices;

---

C++ Code 16.5: Generic routine for pricing European options

```cpp
#include <cmath>
#include <vector>
using namespace std;
#include "fin_recipes.h"

double derivative_price_simulate_european_option_generic(const double& S,
    const double& K,
    const double& r,
    const double& sigma,
    const double& time,
    double payoff(const vector<double>& prices, const double& X),
    const int& no_steps,
    const int& no_sims) {
    double sum_payoffs=0;
    for (int n=0; n<no_sims; n++) {
        vector<double> prices = simulate_lognormally_distributed_sequence(S,r,sigma,time,no_steps);
        sum_payoffs += payoff(prices,K);
    }
    return exp(-r*time) * (sum_payoffs/no_sims);
}
```

C++ Code 16.5: Generic routine for pricing European options

---

`C++ Code 16.4: Simulating a sequence of lognormally distributed variables`
#include <cmath>
#include <numeric>
#include <vector>

using namespace std;

double payoff_arithmetic_average_call(const vector<double>& prices, const double& K) {
    double sum = accumulate(prices.begin(), prices.end(), 0.0);
    double avg = sum / double(prices.size());
    return max(0.0, avg - K);
}

double payoff_geometric_average_call(const vector<double>& prices, const double& K) {
    double logsum = log(prices[0]);
    for (unsigned i = 1; i < prices.size(); ++i) { logsum += log(prices[i]); }
    double avg = exp(logsum / double(prices.size()));
    return max(0.0, avg - K);
}

C++ Code 16.6: Payoff function for Asian call option

#include <vector>
#include <algorithm>

using namespace std;

double payoff_lookback_call(const vector<double>& prices, const double& unused_variable) {
    double m = *min_element(prices.begin(), prices.end());
    return prices.back() - m; // always positive or zero
}

double payoff_lookback_put(const vector<double>& prices, const double& unused_variable) {
    double m = *max_element(prices.begin(), prices.end());
    return m - prices.back(); // max is always larger or equal.
}

C++ Code 16.7: Payoff function for lookback option
16.5 Control variate

As discussed in chapter 14, a control variate is a price which we both have an analytical solution of and find the Monte Carlo price of. The differences between these two prices is a measure of the bias in the Monte Carlo estimate, and is used to adjust the Monte Carlo estimate of other derivatives priced using the same random sequence.

C++ Code 16.8 shows the Black Scholes price used as a control variate. An alternative could have been the analytical lookback price, or the analytical solution for a geometric average price call shown earlier.

```cpp
#include "fin_recipes.h"
#include <cmath>
using namespace std;

double derivative_price_simulate_european_option_generic_with_control_variate(const double& S,
                                const double& K,
                                const double& r,
                                const double& sigma,
                                const double& time,
                                double payoff(const vector<double>& prices,
                                               const double& X),
                                const int& no_steps,
                                const int& no_sims) {
    double c_bs = option_price_call_black_scholes(S,S,sigma,time); // price an at the money Black Scholes call
    double sum_payoffs=0;
    double sum_payoffs_bs=0;
    for (int n=0; n<no_sims; n++) {
        vector<double> prices = simulate_lognormally_distributed_sequence(S,r,sigma,time, no_steps);
        double S1 = prices.back();
        sum_payoffs += payoff(prices,K);
        sum_payoffs_bs += payoff_call(S1,S); // simulate at the money Black Scholes price
    }
    double c_sim = exp(-r*time) * (sum_payoffs/no_sims);
    double c_bs_sim = exp(-r*time) * (sum_payoffs_bs/no_sims);
    c_sim += (c_bs-c_bs_sim);
    return c_sim;
}
```

**C++ Code 16.8: Control Variate**

Example

Using the parameters $S = 100$, $K = 120$, $r = 0.10$, time = 1.0, $\sigma = 0.25$, no sims = 10000, no steps = 250, $q = 0$, price arithmetic and geometric average calls by generic simulation.
C++ program:

```cpp
cout << "Testing general simulation of European options " << endl;
double S=100; double K=120; double r = 0.10;
double time = 1.0; double sigma = 0.25; int no_sims = 10000; int no_steps = 250;
double q=0;
cout << "simulated arithmetic average "
<< " S= " << S << " r= " << r << " price="
<< derivative_price_simulate_european_option_generic(S,K,r,sigma,time,
   payoff_arithmetic_average_call,
   no_steps,no_sims)
<< endl;
cout << "simulated geometric average = 
<< derivative_price_simulate_european_option_generic(S,K,r,sigma,time,
   payoff_geometric_average_call,
   no_steps,no_sims)
<< endl;
cout << "analytical lookback put = 
<< option_price_european_lookback_put(S,S,r,q,sigma,time)
<< endl;
cout << "simulated lookback put = 
<< derivative_price_simulate_european_option_generic(S,0,r,sigma,time,
   payoff_lookback_put,
   no_steps,no_sims)
<< endl;
cout << "analytical lookback call = 
<< option_price_european_lookback_call(S,S,r,q,sigma,time)
<< endl;
cout << "simulated lookback call = 
<< derivative_price_simulate_european_option_generic(S,0,r,sigma,time,
   payoff_lookback_call,
   no_steps,no_sims)
<< endl;
cout << "simulated lookback call using control variates = 
<< derivative_price_simulate_european_option_generic_with_control_variate(S,0,r,sigma,time,
   payoff_lookback_call,
   no_steps,no_sims)
<< endl;
```

Output from C++ program:

Testing general simulation of European options
simulated arithmetic average  S= 100  r= 0.1 price=1.49696
simulated geometric average = 1.38017
analytical lookback put = 16.2665
simulated lookback put = 14.9846
analytical lookback call = 22.8089
simulated lookback call = 21.9336
simulated lookback call using control variates = 22.0685

16.6 References

Chapter 17

Generic binomial pricing

Contents

17.1 Introduction ................................................................. 177
17.2 Delta calculation ............................................................ 182

17.1 Introduction

In earlier chapters we have seen a large number of different versions of the binomial pricing formula. In this chapter we see how we can build a framework for binomial pricing that lets us write a single generic routine that can be used for a binomial approximation of all sorts of derivatives. The important feature that lets us write such a generic routine is that the only place the terms of the derivative appears in the calculation is the calculation of the value at each node. Consider the binomial approximation of an American call in Chapter 12’s C++ Code 12.2, repeated below as C++ Code 17.1 for convenience.

The terms of the derivative only appears when calculating the value at the nodes, where the calculation \( \max(\text{prices}[]-K,0) \) appears. The key to building a generic binomial routine lies in replacing this calculation with a generic routine. C++ Code 17.2 shows how the generic routine is implemented.
double option_price_call_american_binomial(const double& S, // spot price
                                         const double& X, // exercise price
                                         const double& r, // interest rate
                                         const double& sigma, // volatility
                                         const double& t, // time to maturity
                                         const int& steps) { // no steps in binomial tree
    double R = exp(r*(t/steps)); // interest rate for each step
    double Rinv = 1.0/R; // inverse of interest rate
    double u = exp(sigma*sqrt(t/steps)); // up movement
    double d = 1.0/u;
    double p_up = (R-d)/(u-d); // up movement
    double p_down = 1.0-p_up;

    vector<double> prices(steps+1); // price of underlying
    prices[0] = S*pow(d, steps); // fill in the endnodes.
    double uu = u*u;
    for (int i=1; i<=steps; ++i) prices[i] = uu*prices[i-1];

    vector<double> call_values(steps+1); // value of corresponding call
    for (int i=0; i<=steps; ++i) call_values[i] = max(0.0, (prices[i]-X)); // call payoffs at maturity

    for (int step=steps-1; step>=0; --step) {
        for (int i=0; i<=step; ++i) {
            call_values[i] = (p_up*call_values[i+1]+p_down*call_values[i])*Rinv;
            prices[i] = d*prices[i+1];
            call_values[i] = max(call_values[i], prices[i]-X); // check for exercise
        }
    }
    return call_values[0];
}

C++ Code 17.1: Binomial price of American Call
```cpp
#include <cmath> // standard mathematical library
#include <algorithm> // defines the max() operator
#include <vector> // STL vector templates
using namespace std;

double option_price_generic_binomial(const double& S, const double& K, double generic_payoff(const double& S, const double& K), const double& t, const double& sigma, const double& r, const double& t, const int& steps) {
    double R = exp(r*t/steps); // interest rate for each step
    double Rinv = 1.0/R; // inverse of interest rate
    double u = exp(sigma*sqrt(t/steps)); // up movement
    double d = 1.0/u;
    double p_up = (R*d)/u;
    double p_down = (R-d)/(u-d);
    vector<double> prices(steps+1); // price of underlying
    prices[0] = S*pow(d, steps); // fill in the endnodes.
    double uu = u*u;
    for (int i=1; i<=steps; ++i) prices[i] = uu*prices[i-1];
    vector<double> values(steps+1); // value of corresponding call
    for (int i=0; i<=steps; ++i) values[i] = generic_payoff(prices[i], K); // payoffs at maturity
    for (int step=steps-1; step>=0; --step) {
        for (int i=0; i<=step; ++i) {
            values[i] = (p_up*values[i+1]+p_down*values[i])*Rinv; // value by not exercising
            prices[i] = d*prices[i+1];
            values[i] = max(values[i], generic_payoff(prices[i], K)); // check for exercise
        }
    }
    return values[0];
}
```

C++ Code 17.2: Generic binomial calculation
Using this routine is then merely a matter of providing a definition of the derivative payoff. **C++ Code 17.3** shows how the payoffs are defined for standard put and call options. Pricing American put and call options is then merely a matter of supplying these payoff definitions to the generic binomial routine.

```cpp
#include <algorithm>
using namespace std;

double payoff_call(const double & S, const double & K) {
    return max(0.0, S - K);
};
double payoff_put (const double & S, const double & K) {
    return max(0.0, K - S);
};
```

**C++ Code 17.3: Payoff definitions for put and call options**

**Example**

Consider American call and put option on non-dividend paying stock, where \( S = 100, K = 100, \sigma = 0.2, (T - t) = 1 \) and \( r = 0.1 \).

1. Price the options using binomial approximations with 100 steps.

```cpp
double S = 100.0;
double K = 100.0;
double r = 0.1;
double sigma = 0.25;
double time_to_maturity=1.0;
int steps = 100;
cout << " american call price = "
 << option_price_generic_binomial(S,K,payoff_call, r, sigma, time_to_maturity, steps)
 << endl;
cout << " american put price = "
 << option_price_generic_binomial(S,K,payoff_put, r, sigma, time_to_maturity, steps)
 << endl;
```

**Output from C++ program:**

American call price = 14.9505
American put price = 6.54691

More exotic options can also be calculated using the same approach. For example, **C++ Code 17.4** shows payoff definitions for two binary options, options that pay off one dollar if the price of the underlying is above \( K \) (call) or below \( K \) (put). The typical such option is European, the check for whether the option is in the money happens at maturity, but one can also think of cases where the binary option pays off one dollar as soon as \( S \), the price of the underlying, hits the “barrier” \( K \).

**Example**

Consider binary options that pays one dollar if the price \( S \) of the underlying increases to \( K \) or above during some time period. Suppose the current price of the underlying is \( S = 100 \). Using \( \sigma = 0.2, (T - t) = 1 \) and \( r = 0.1 \), price such a binary option when \( K = 120 \).
double payoff_binary_call(const double& S, const double& K){
    if (S>=K) return 1;
    return 0;
};

double payoff_binary_put(const double& S, const double& K){
    if (S<=K) return 1;
    return 0;
};

C++ Code 17.4: Payoff definitions for binomial options

C++ program:

    double S = 100.0;
    double K = 120.0;
    double r = 0.1;
    double sigma = 0.25;
    double time_to_maturity=1.0;
    int steps = 100;
    cout << " binary option price = "
         << option_price_generic_binomial(S,K,payoff_binary_call, r, sigma, time_to_maturity, steps)
         << endl;

Output from C++ program:

    binary option price = 0.498858
17.2 Delta calculation

Deltas and other greeks are calculated using the same style of generic routine. C++ Code 17.5 shows how to calculate delta using the same generic approach.

C++ Code 17.5: Generic binomial calculation of delta

Exercise 17.1.

The generic routines discussed in this chapter have been for American options. How would you modify them to account for European options?
Chapter 18

Trinomial trees

Contents

18.1 Intro ......................................................... 183
18.2 Implementation ............................................ 183
18.3 Further reading ............................................. 185

18.1 Intro

A trinomial tree is similar to a binomial tree, just with one more branch. At each point of time there are three possible future values of the underlying $S$.

\[ S_u = uS_0 \]
\[ S_m = S_0 \]
\[ S_d = dS_0 \]

18.2 Implementation

A trinomial tree can be implemented with various parameterizations. We will show the calculation using the following parameterization:

\[ u = e^{\sigma \sqrt{3} \Delta t} \]
\[ e = \frac{1}{u} \]
\[ p_u = -\sqrt{\frac{\Delta t}{12\sigma^2}} \left(r - q - \frac{\sigma^2}{2}\right) + \frac{1}{6} \]
\[ p_m = \frac{2}{3} \]
\[ p_u = \sqrt{\frac{\Delta t}{12\sigma^2}} \left(r - q - \frac{\sigma^2}{2}\right) + \frac{1}{6} \]
We calculate the option price using the usual roll back procedure with the continuation value

\[ e^{-r\Delta t} (p_u f_u + p_m f_m + p_d f_d) \]

```cpp
#include <vector>
#include <cmath>
using namespace std;

double option_price_put_american_trinomial( const double& S,  
                                          const double& K,  
                                          const double& r,  
                                          const double& q,  
                                          const double& sigma, 
                                          const double& t,    
                                          const int& steps) {
  double delta_t = t/steps;
  double Rinv = exp(-r*(delta_t));
  double sigma_sqr = pow(sigma,2);
  double u = exp(sigma*sqrt(3.0*delta_t));
  double d = 1.0/u;
  double p_u = 1.0/6.0 + sqrt(delta_t/(12.0*sigma_sqr)) * (r-q-0.5*sigma_sqr);
  double p_m = 2.0/3.0;
  double p_d = 1.0/6.0 - sqrt(delta_t/(12.0*sigma_sqr)) * (r-q-0.5*sigma_sqr);

  vector<vector<double>> Stree;  // price of underlying in a tree
  vector<double> Svec;
  Svec.push_back(S);
  for (int step=1; step<=steps; ++step) {
    Stree.push_back(Svec);
    Svec.insert(Svec.begin(),Svec[0]*d); // use the fact that only the extreme values change.
    Svec.push_back(Svec[Svec.size()-1]*u);
  }
  int m = int(Svec.size());
  vector<double> values_next = vector<double>(m);  // value of option next step
  for (int i=0; i< m; ++i) values_next[i] = max(0.0, K-Svec[i]);  // call payoffs at maturity
  vector<double> values;
  for (int step=steps-1; step>=0; --step) {
    m = int(Stree[step].size());
    values = vector<double>(m);  // value of option
    for (int i=0; i< m; ++i) {
      values[i] = (p_u*values_next[i+2]+p_m*values_next[i+1] + p_d*values_next[i])*Rinv;
      values[i] = max(values[i],K-Stree[step][i]);  // check for exercise
    }
    values_next=values;
  }
  return values[0];
}
```

C++ Code 18.1: Price of american put using a trinomial tree

**Example**

You are given the following information about an option: \( S = 100, K = 100, r = 0.1, \sigma = 0.25 \) and time to maturity is 1 year. Price an American put using a trinomial approximation with 100 steps.
function P = opt_price_trinom_am_put(S, K, r, q, sigma, t, steps)
    delta_t = t/steps;
    sigma_sqr = sigma^2;
    Rinv = exp(-r*delta_t);
    u = exp(sigma*sqrt(3.0*delta_t));
    d = 1.0/u;
    p_u = 1/6 + sqrt(delta_t/(12*sigma_sqr)) * (r-q-0.5*sigma_sqr);
    p_m = 2/3;
    p_d = 1/6 - sqrt(delta_t/(12*sigma_sqr)) * (r-q-0.5*sigma_sqr);

    Svec = [ S ];
    Stree = [ Svec ];
    Su = S;
    Sd = S;
    for step = 1:steps+1
        Su = Su*u;
        Sd = Sd*d;
        Svec = [ Sd; Svec; Su ];
        Stree = [ Stree; zeros(2,step) Svec ];
    end
    values_next = max(0,K-Svec);
    for step = steps:-1:1
        m = 2*step+1;
        values = Rinv*(p_u*values_next(3:m+2)+p_m*values_next(2:m+1)+p_d*values_next(1:m));
        values = max(values, K-Stree(1:m,step+1));
        values_next = values;
    end
    P = values;
end

Matlab Code 18.1: Price of american put using a trinomial tree

C++ program:

```cpp
double S = 100.0; double K = 100.0;
double r = 0.1; double q = 0; double sigma = 0.25;
int no_steps = 100;
cout << " american put = " << option_price_put_american_trinomial(S,K,r,q,sigma,t,0.1,no_steps) << endl;
```

Output from C++ program:

```
ablean put = 6.52719
```

Exercise 18.1.

In the code for the trinomial tree the price of the underlying is put into a triangular data structure. This can be collapsed into a single array, since the only changes when you add one step is to add one price at the top and at the bottom. If you keep track of the top and bottom of the current array you can access the prices of the underlying through some clever indexing into the single vector of prices of the underlying.

Exercise 18.2.

Similarly to the binomial case, one can build a generic trinomial tree where the payoff function is passed as a parameter to the routine. Implement such a generic trinomial tree.

18.3 Further reading

Hull (2011)
Chapter 19

Alternatives to the Black Scholes type option formula

Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>19.1</td>
<td>Merton’s Jump diffusion model.</td>
<td>186</td>
</tr>
<tr>
<td>19.2</td>
<td>Heston’s pricing formula for a stochastic volatility model</td>
<td>188</td>
</tr>
</tbody>
</table>

A large number of alternative formulations to the Black Scholes analysis has been proposed. Few of them have seen any widespread use, but we will look at some of these alternatives.

19.1 Merton’s Jump diffusion model.

Merton (1976) has proposed a model where in addition to a Brownian Motion term, the price process of the underlying is allowed to have jumps. The risk of these jumps is assumed to not be priced.

In the following we look at an implementation of a special case of Merton’s model, described in (Hull, 1993, pg 454), where the size of the jump has a normal distribution. \( \lambda \) and \( \kappa \) are parameters of the jump distribution. The price of an European call option is given in Formula 19.1 and implemented in C++ Code 19.1.

\[
c = \sum_{n=0}^{\infty} \frac{e^{\lambda' \tau}(\lambda' \tau)^n}{n!} C_{BS}(S, X, \tau_n, \sigma_n^2, T - t)
\]

where

\[
\tau = T - t
\]

\[
\lambda' = \lambda(1 + \kappa)
\]

\( C_{BS}(\cdot) \) is the Black Scholes formula, and

\[
\sigma_n^2 = \sigma^2 + \frac{n \delta^2}{\tau}
\]

\[
\tau_n = \tau - \lambda \kappa + \frac{n \ln(1 + \kappa)}{\tau}
\]

Formula 19.1: The option pricing formula of the Merton (1976) model

In implementing this formula, we need to terminate the infinite sum at some point. But since the
factorial function is growing at a much higher rate than any other, that is no problem, terminating at
about \( n = 50 \) should be on the conservative side. To avoid numerical difficulties, use the following
method for calculation of

\[
\frac{e^{X'}(\lambda'\tau)^n}{n!} = e^{\ln \left( \frac{e^{X'}(\lambda'\tau)^n}{n!} \right)} = e^{\lambda'\tau + n \ln(\lambda'\tau) - \sum_{i=1}^{n} \ln i}
\]

```cpp
#include <cmath>
#include "fin_recipes.h"

double option_price_call_merton_jump_diffusion(const double& S, const double& X, const double& r, const double& sigma, const double& time_to_maturity, const double& lambda, const double& kappa, const double& delta) {
    const int MAXN = 50;
    double tau = time_to_maturity;
    double sigma_sqr = sigma*sigma;
    double delta_sqr = delta*delta;
    double lambdaPrime = lambda * (1+kappa);
    double gamma = log(1+kappa);
    double c = exp(-lambdaPrime*tau)*option_price_call_black_scholes(S,X,r-lambda*kappa,sigma,tau);
    double log_n = 0;
    for (int n=1; n<MAXN; ++n) {
        log_n += log(double(n));
        double sigma_n = sqrt(sigma_sqr+n*delta_sqr/tau);
        double r_n = r-lambda*kappa+n*gamma/tau;
        c += exp(-lambdaPrime*tau+n*log(lambdaPrime*tau)-log_n)*
            option_price_call_black_scholes(S,X,r_n.sigma_n.n_tau);
    }
    return c;
}
```

C++ Code 19.1: Merton's jump diffusion formula

Example

Price an option using Merton's jump diffusion formula, using the parameters
\( S = 100, K = 100, r = 0.05, \sigma = 0.3, \lambda = 0.5, \kappa = 0.5, \delta = 0.5 \).

C++ program:

```cpp
double S=100; double K=100; double r=0.05;
double sigma=0.3; double time_to_maturity=1;
double lambda=0.5; double kappa=0.5; double delta=0.5;
cout << " Merton Jump diffusion call = " <<
    option_price_call_merton_jump_diffusion(S,K,r,sigma,time_to_maturity,lambda,kappa,delta)
<< endl;
```

Output from C++ program:

Merton Jump diffusion call = 23.2074
Heston (1993) relaxes the Black-Scholes assumption of a constant volatility by introducing a stochastic volatility. He finds exact solutions for European options. Let $S$ be the stock price and $v$ the volatility. These two variables are assumed to follow joint stochastic processes.

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= \mu dt + \sqrt{v(t)} S(t) dz_1(t) \\
\frac{dv(t)}{v(t)} &= -\kappa \frac{v(t)}{v_0} dt + \sigma \sqrt{v(t)} dz_2(t)
\end{align*}
\]

The two processes $z_1$ and $z_2$ have correlation $\rho$. Rewrite the process for the volatility as

\[
\frac{dv(t)}{v(t)} = \left(\kappa (\phi - v(t)) + \theta \right) dt + \sigma \sqrt{v(t)} dz_2(t)
\]

Let $\lambda(S, v, t)$ be the price of volatility risk. Under a constant interest rate $r$

\[
P(t, T) = e^{-r(T-t)}
\]

Consider a call option with strike price $K$. Its price is given by Formula 19.2.

\[
C(s, v, t) = SP_1 - K P(t, T) P_2
\]

where

\[
P_j(x, v, T, \ln(K)) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \Re \left[ e^{-i\phi \ln(K)} f_j(x, v, T, \phi) \frac{d\phi}{i\phi} \right]
\]

\[
f_j(x, v, T, \phi) = e^{C(\tau, \phi) + D(\tau, \phi) v + i\phi \sigma}
\]

\[
C(\tau, \phi) = \tau \phi i\tau + \frac{a}{\sigma^2} \left( (b_1 - \rho \sigma \phi \phi + d) \tau - 2 \ln \left[ \frac{1 - e^{d\tau}}{1 - g} \right] \right)
\]

\[
D(\tau, \phi) = \frac{b_j - \rho \sigma \phi \phi + d}{\sigma^2} \left[ \frac{1 - e^{d\tau}}{1 - g e^{d\tau}} \right]
\]

\[
u_1 = \frac{1}{2}, \quad \nu_2 = -\frac{1}{2}
\]

\[
a = \kappa \theta
\]

\[
b_1 = \kappa + \lambda - \rho \sigma \quad b_2 = \kappa + \lambda
\]

\[
x = \ln(S)
\]

\[
g = \frac{b_j - \rho \sigma \phi \phi + d}{b_j - \rho \sigma \phi \phi - d}
\]

\[
d = \sqrt{\left(\rho \sigma \phi \phi - b_j\right)^2 - \sigma^2 \left(2u_j \phi \phi - \phi^2\right)}
\]


The implementation of this pricing formula has some instructive C++ features. First, it illustrates calculations of of complex variables. Complex numbers is part of the C++ standard, and are accessed by including the

```cpp
#include <complex>
```
statement. Complex numbers are templated, it is necessary to specify what type of floating point type to use, such as `complex<double>` or `complex<double double>`.

To evaluate the price it is also necessary to do a numerical integration. In the calculation this is solved by a call to an external routine. We use a routine provided by the Gnu GSL project.\(^1\)

**Example**

Given the following set of parameters: \( S = 100, K = 100, r = 0.01, \nu = 0.01, \tau = 0.5, \rho = 0, \kappa = 2, \lambda = 0, \theta = 0.01 \) and \( \sigma = 0.01 \), price a call option using the Heston formula

\[
\text{C++ program:}
\]

```c++
double S=100; double K=100; double r=0.01; double nu=0.01;
double tau=0.5; double rho=0; double kappa=2; double lambda=0.0;
double theta=0.01; double sigma=0.01;
cout << "heston call price "
    << heston_call_option_price( S, K, r, nu, tau, rho, kappa, lambda, theta, sigma) << endl;
```

**Output from C++ program:**

```
heston call price 3.06944
```

\(^1\) An example alternatives would have been Numerical Recipes in C++'s quadrature.
#include <iostream>
#include <cmath>
#include <complex>
using namespace std;
#include "gsl/gsl_integration.h"

struct heston_Parms
{
  double K;
  double x;
  double r;
  double v;
  double tau;
  double kappa;
  double theta;
  double rho;
  double sigma;
  double lambda;
  int j;
};

extern "C"{
  double heston_integrand_j(double phi, void *p){
    struct heston_Parms* parms = (struct heston_Parms*)p;
    double K = (parms->K);
    double x = (parms->x);
    double v = (parms->v);
    double r = (parms->r);
    double kappa = (parms->kappa);
    double theta = (parms->theta);
    double rho = (parms->rho);
    double sigma = (parms->sigma);
    double lambda = (parms->lambda);
    double tau = (parms->tau);
    double j = (parms->j);
    double sigma_sqr = pow(sigma,2);
    double uj; double bj;
    if (j==1){ uj=0.5; bj=kappa+lambda-rho*sigma; }
    else { uj=-0.5; bj=kappa+lambda; }
    complex<double> i(0,1);
    double a = kappa*theta;
    complex<double> d = sqrt( pow(rho*sigma*phi*i-bj,2) - sigma_sqr*(2*uj*phi*i-pow(phi,2)) );
    complex<double> g = (bj - rho*sigma*phi*i+d)/(bj-rho*sigma*phi*i-d);
    complex<double> C = r*phi*i*tau + (a/sigma_sqr)*((bj-rho*sigma*phi*i+d)*tau - 2.0*log((1.0-g*exp(d*tau))/(1.0-g)));
    complex<double> D = (bj-rho*sigma*phi*i+d)/sigma_sqr * ( (1.0-exp(d*tau))/(1.0-g*exp(d*tau)) );
    complex<double> f1 = exp(C+D*v+i*phi*x);
    complex<double> F = exp(-phi*i*log(K))*f1/(i*phi);
    return real(F);
  }
};

inline double heston_Pj(double S, double K, double r, double v, double tau, double sigma, double kappa, double lambda, double rho, double theta, int j){
  double x=log(S);
  struct heston_Parms parms = { K, x, r, v, tau, kappa, theta, rho, sigma, lambda, j};
  size_t n=10000;
  gsl_integration_workspace* w = gsl_integration_workspace_alloc(n);
  gsl_function F;
  F.function = &heston_integrand_j;
  F.params = &parms;
  double result, error;
  gsl_integration_qagiu(&F,0.1e-7,1e-7,n,w,&result,&error); // integral to infinity starting at zero
  return 0.5 + result/M_PI1;
};

double heston_call_option_price(const double& S, const double& K, const double& r, const double& v, const double& tau, const double& rho, const double& kappa, const double& lambda, const double& theta, const double& sigma){
  double P1 = heston_Pj(S,K,r,v,tau,sigma,kappa,lambda,rho,theta,1);
  double P2 = heston_Pj(S,K,r,v,tau,sigma,kappa,lambda,rho,theta,2);
  double C=S*P1-K*exp(-r*tau)*P2;
  return C;
};

C++ Code 19.2: Hestons pricing formula for a stochastic volatility model
Chapter 20

Pricing of bond options, basic models

Contents

20.1 Black Scholes bond option pricing ................................. 191
20.2 Binomial bond option pricing ....................................... 193

The area of fixed income securities is one where a lot of work is being done in creating advanced mathematical models for pricing of financial securities, in particular fixed income derivatives. The focus of the modelling in this area is on modelling the term structure of interest rates and its evolution over time, which is then used to price both bonds and fixed income derivatives. However, in some cases one does not need the machinery of term structure modelling which we’ll look at in later chapters, and price derivatives by modelling the evolution of the bond price directly.

Specifically, suppose that the price of a Bond follows a Geometric Brownian Motion process, just like the case we have studied before. This is not a particularly realistic assumption for the long term behaviour of bond prices, since any bond price converges to the bond face value at the maturity of the bond. The Geometric Brownian motion may be OK for the case of short term options on long term bonds.

20.1 Black Scholes bond option pricing

Given the assumed Brownian Motion process, prices of European Bond Options can be found using the usual Black Scholes formula, as shown in C++ Code 20.1 for a zero coupon bond and C++ Code 20.2 for the case of an option on a coupon bond.

```cpp
#include <cmath>
#include "normdist.h"

double bond_option_price_put_zero_black_scholes(const double& B,
                                             const double& X,
                                             const double& r,
                                             const double& sigma,
                                             const double& time){

double time_sqrt = sqrt(time);
    double d1 = (log(B/X)+r*time)/(sigma*time_sqrt) + 0.5*sigma*time_sqrt;
    double d2 = d1-(sigma*time_sqrt);
    double p = X * exp(-r*time) * N(-d2) - B * N(-d1);
    return p;
}
```

C++ Code 20.1: Black scholes price for European put option on zero coupon bond
#include <cmath>
#include <vector>
using namespace std;
#include "normdist.h"
#include "fin_recipes.h"

double bond_option_price_put_coupon_bond_black_scholes( const double& B,
const double& X,
const double& r,
const double& sigma,
const double& time,
const vector<double> coupon_times,
const vector<double> coupon_amounts){
    double adjusted_B=B;
    for (unsigned int i=0;i<coupon_times.size();i++) {
        if (coupon_times[i]<=time) {
            adjusted_B -= coupon_amounts[i] * exp(-r*coupon_times[i]);
        }
    }
    return bond_option_price_put_zero_black_scholes(adjusted_B,X,r,sigma,time);
};

C++ Code 20.2: Black scholes price for European put option on coupon bond
20.2 Binomial bond option pricing

Since we are in the case of geometric Brownian motion, the usual binomial approximation can be used to price American options, where the bond is the underlying security. C++ Code 20.3 shows the calculation of a put price

```c++
#include <cmath> // standard mathematical library
#include <algorithm> // defining the max() operator
#include <vector> // STL vector templates
using namespace std;

double bond_option_price_put_american_binomial(const double& B, // Bond price
                                               const double& K, // exercise price
                                               const double& r, // interest rate
                                               const double& sigma, // volatility
                                               const double& t, // time to maturity
                                               const int& steps){ // no steps in binomial tree
	double R = exp(r*t/steps); // interest rate for each step
double Rinv = 1.0/R; // inverse of interest rate
double u = exp(sigma*sqrt(t/steps)); // up movement
double uu = u*u;
double d = 1.0/u;
double p_up = (R-d)/(u-d);
double p_down = 1.0-p_up;
vector<double> prices(steps+1); // price of underlying
vector<double> put_values(steps+1); // value of corresponding put

prices[0] = B*pow(d, steps); // fill in the endnodes.
for (int i=1; i<=steps; ++i) prices[i] = uu*prices[i-1];
for (int i=0; i<=steps; ++i) put_values[i] = max(0.0, (K-prices[i])); // put payoffs at maturity
for (int step=steps-1; step>=0; --step) {
    for (int i=0; i<=step; ++i) {
        put_values[i] = (p_up*put_values[i+1]+p_down*put_values[i])*Rinv;
        prices[i] = d*prices[i+1];
        put_values[i] = max(put_values[i], (K-prices[i])); // check for exercise
    }
}
return put_values[0];
```

C++ Code 20.3: Binomial approximation to american put bond option price
Example
Parameters: $B = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.1$, $\text{time}=1$.

There is also a coupon bond with the same bond price, but paying coupon of 0.5 at date 1.

1. Price a an European put option on the zero coupon bond using Black Scholes.

2. Price a an European put option on the coupon coupon bond using Black Scholes.

3. Price a an European put option on the zero coupon bond using binomial approximation with 100 steps.

C++ program:

```cpp
double B=100;
double K=100;
double r=0.05;
double sigma=0.1;
double time=1;
cout << " zero coupon put option price = "
    << bond_option_price_put_zero_black_scholes(B,K,r,sigma,time) << endl;

vector<double> coupon_times; coupon_times.push_back(0.5);
vector<double> coupons; coupons.push_back(1);
cout << " coupon bond put option price = "
    << bond_option_price_put_coupon_bond_black_scholes(B,K,r,sigma,time,coupon_times,coupons) << endl;

int steps=100;
cout << " zero coupon american put option price, binomial = "
    << bond_option_price_put_american_binomial(B,K,r,sigma,time,steps) << endl;
```

Output from C++ program:

zero coupon put option price = 1.92791
coupon bond put option price = 2.22852
zero coupon american put option price, binomial = 2.43282
Chapter 21

Credit risk

Contents

21.1 The Merton Model .......................................................... 195
21.2 Issues in implementation ............................................... 196

Option pricing has obvious applications to the pricing of risky bonds.

21.1 The Merton Model

This builds on the Black and Scholes (1973) and Merton (1973) framework to find the value of the debt issued by the firm. The ideas were already in Black and Scholes, who discussed the view of the firm as a call option.

Assume debt structure: There is a single debt issue. Debt is issued as a zero coupon bond. The bond is due on a given date $T$.

Assuming the firm value $V$ follows the usual Brownian motion process, debt is found as a closed form solution, similar in structure to the Black Scholes equation for a call option.

Easiest seen from the interpretation of firm debt as the price of risk free debt, minus the value of a put option.

Price debt by the price $B$ of risk free debt, and then subtract the price of the put, using the Black Scholes formula.

The Black Scholes formula for a call option is

$$c = S \cdot N(d_1) - K \cdot e^{-r(T-t)} N(d_2)$$

where

$$d_1 = \frac{\ln \left( \frac{S}{K} \right) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}$$

$$d_2 = d_1 - \sigma \sqrt{T-t}$$

$N(\cdot)$ = The cumulative normal distribution

$$p = K e^{-r(T-t)} N(-d_2) - S N(-d_1)$$

In the context here, reinterpret $S$ as $V$, firm value. The put is priced as

$$p = K e^{-r(T-t)} N(-d_2) - V_i N(-d_1)$$

195
where
\[ d_1 = \frac{\ln \left( \frac{V}{F} \right) + (r + \frac{1}{2} \sigma^2)(T - t)}{\sigma \sqrt{T - t}} \]

Note on interpretation: The spread between risky and risk free debt determined solely by the price of the put option.

**Example**
The current value of the firm \( V = 100 \). The firm has issued one bond with face value 90, which is due to be paid one year from now. The risk free interest rate is 5% and the volatility of the firms value is 25%. Determine the value of the debt.

**C++ program:**
```cpp
cout << " Credit Risk Calculation " << endl;
double V=100; double F=90; double r=0.05; double T=1; double sigma=0.25;
double p = option_price_put_black_scholes(V,F,r,sigma,T);
cout << " Debt value = " << exp(-r*T)*F - p << endl;
```

**Output from C++ program:**
```
Credit Risk Calculation
Debt value = 81.8592
```

### 21.2 Issues in implementation

- Firm value and firm volatility is unobservable.
- The model assumes a simple debt structure, most debt structures tend to be more complex.
Chapter 22

Term Structure Models

Contents

22.1 The Nelson Siegel term structure approximation ................................................. 198
22.2 Extended Nelson Siegel models ................................................................. 200
22.3 Cubic spline. ......................................................................................... 202
22.4 Cox Ingersoll Ross. ........................................................................... 205
22.5 Vasicek ................................................................. 208
22.6 Readings ............................................................. 210

We now expand on the analysis of the term structure in chapter 5. As shown there, the term structure is best viewed as an abstract class providing, as functions of term to maturity, the prices of zero coupon bonds (discount factors), yield on zero coupon bonds (spot rates) or forward rates. In the earlier case we considered two particular implementations of the term structure: A flat term structure or a term structure estimated by linear interpolations of spot rates. We now consider a number of alternative term structure models. The focus of this chapter is empirical, we consider ways in which one can specify a term structure in a lower dimensional way. Essentially we are looking at ways of doing curve-fitting, of estimating a nonlinear relationship between time and discount factors, or between time and spot rates. Since the relationship is nonlinear, this is a nontrivial problem. One has to choose a functional form to estimate, which allows enough flexibility to “fit” the term structure, but not so flexible that it violates the economic restrictions on the term structure. Here are some considerations.

- Discount factors must be positive. \( (d_t > 0) \). This is because they are prices, negative prices allow for arbitrage.
- Discount factors must be a nonincreasing function of time. \( (d_t \geq d_{t+k} \forall k > 0) \). Again, this is to avoid arbitrage.
- Nominal interest rates can not be negative. \( (r_t \geq 0 \forall t) \) This is another implication of the absence of arbitrage opportunities.
- Both discount factors and interest rates must be smooth functions of time.
- The value of a payment today is the payment today. \( d_0 = 1 \).

A number of alternative ways of estimating the term structure has been considered. Some are purely used as interpolation functions, while others are fully specified, dynamic term structure models. Of the models that follow, the approximating function proposed in Nelson and Siegel (1987) and the cubic spline used by e.g. McCulloch (1971) are examples of the first kind, and the term structure models of Cox, Ingersoll, and Ross (1985) and Vasicek (1977) are examples of the second kind.
What is the typical use of the functions we consider here? One starts with a set of fixed income securities, typically a set of treasury bonds. Observing the prices of these bonds, one asks: What set of discount factors is most likely to have generated the observed prices. Or: What term structure approximations provides the “best fit” to this set of observed bond prices.

22.1 The Nelson Siegel term structure approximation

Nelson and Siegel (1987) proposes the parameterization shown in Formula Refns

\[ r(t) = \beta_0 + (\beta_1 + \beta_2) \left[ \frac{1 - e^{-\frac{t}{\lambda}}}{\frac{t}{\lambda}} \right] + \beta_2 \left[ e^{-\frac{t}{\lambda}} \right] \]

Notation: \( t \): Time to maturity. \( r \): spot interest rate. \( \beta_0, \beta_1, \beta_2 \) and \( \lambda \): constants.

**Formula 22.1**: Nelson and Siegel (1987) parameterization of term structure

The implementation of this calculation is shown in **C++ Code 22.1**.

```cpp
#include <cmath>
using namespace std;

double term_structure_yield_nelson_siegel(const double& t, const double& beta0, const double& beta1, const double& beta2, const double& lambda) {
    if (t==0.0) return beta0;
    double tl = t/lambda;
    double r = beta0 + (beta1+beta2) * ((1-exp(-tl))/tl) + beta2 * exp(-tl);
    return r;
}
```

**C++ Code 22.1**: Calculation of the Nelson and Siegel (1987) term structure model

This is wrapped in a term structure class as shown in **Header File 22.1** and **C++ Code 22.2**.

```cpp
class term_structure_class_nelson_siegel : public term_structure_class {
private:
    double beta0_, beta1_, beta2_, lambda_;
public:
    term_structure_class_nelson_siegel(const double& beta0, const double& beta1, const double& beta2, const double& lambda);
    virtual double yield(const double& T) const;
};
```

**Header file 22.1**: Header file defining a term structure class wrapper for the Nelson Siegel approximation
```cpp
#include "fin_recipes.h"

term_structure_class::term_structure_class_nelson_siegel(const double & b0,
                                                       const double & b1,
                                                       const double & b2,
                                                       const double & l) {
  beta0_  = b0; beta1_ = b1; beta2_ = b2; lambda_ = l;
}

double term_structure_class::nelson_siegel::r(const double & t) const {
  if (t<=0.0) return beta0_;
  return term_structure::yield_nelson_siegel(t,beta0_,beta1_,beta2_,lambda_);
}

C++ Code 22.2: Defining a term structure class wrapper for the Nelson Siegel approximation
```
Example

Using the parameters $\beta_0 = 0.01$, $\beta_1 = 0.01$, $\beta_2 = 0.01$, $\lambda = 5.0$ and $t = 1$ in the Nelson Siegel approximation, find the 1 year discount factor and spot rate, and the forward rate between years 1 and 2.

**C++ program:**

```cpp
#include <iostream>
#include <cmath>

double beta0 = 0.01; double beta1 = 0.01; double beta2 = 0.01; double lambda = 5.0;
double t = 1.0;

cout << "Example calculations using the Nelson Siegel term structure approximation" << endl;
cout << " direct calculation, yield = "
     << term_structure_yield_nelson_siegel(t, beta0, beta1, beta2, lambda) << endl;
term_structure_class_nelson_siegel ns(beta0, beta1, beta2, lambda);
cout << " using a term structure class" << endl;
cout << " yield (t=1) = " << ns.r(t) << endl;
cout << " discount factor (t=1) = " << ns.d(t) << endl;
cout << " forward rate (t1=1, t2=2) = " << ns.f(1, 2) << endl;
```

**Output from C++ program:**

Example calculations using the Nelson Siegel term structure approximation
    direct calculation, yield = 0.0363142
    using a term structure class
    yield (t=1) = 0.0363142
    discount factor (t=1) = 0.964337
    forward rate (t1=1, t2=2) = 0.0300602

22.2 Extended Nelson Siegel models

The Nelson and Siegel (1987) model is simple, with parameters with clear obvious economic interpretations. It does have the problem that the term structure shapes that it allows is limited. To allow for more complex shapes, such as humped shapes, it has been extended in various ways. A popular approximation was introduced by Lars Svensson, parameterized as shown in [Formula 22.2](#)

\[
    r(t) = \beta_0 + \beta_1 \left( \frac{1 - e^{-\frac{t}{\tau_1}}}{\frac{t}{\tau_1}} \right) + \beta_2 \left( \frac{1 - e^{-\frac{t}{\tau_2}}}{\frac{t}{\tau_2}} - e^{-\frac{t}{\tau_2}} \right) + \beta_3 \left( \frac{1 - e^{-\frac{t}{\tau_3}}}{\frac{t}{\tau_3}} - e^{-\frac{t}{\tau_3}} \right)
\]

Notation: $t$: Time to maturity. $r$: spot interest rate. $\beta_0, \beta_1, \beta_2$ and $\lambda$: constants.

**Formula 22.2:** Svensson’s extension of the Nelson and Siegel (1987) parameterization of term structure

This is wrapped in a term structure class as shown in [Header File 22.2](#) and [C++ Code 22.4](#).
```cpp
#include <cmath>
using namespace std;

double term_structure_yield_svensson(const double& t,
    const double& beta0,
    const double& beta1,
    const double& beta2,
    const double& beta3,
    const double& tau1,
    const double& tau2){
    if (t==0.0) return beta0;
    double r = beta0;
    r += beta1 * ((1-exp(-t/tau1))/(t/tau1));
    r += beta2 * ((1-exp(-t/tau1))/(t/tau1)) - exp(-t/tau1);
    r += beta3 * ((1-exp(-t/tau2))/(t/tau2)) - exp(-t/tau2);
    return r;
};
```


```cpp
class term_structure_class_svensson:public term_structure_class {
private:
    double beta0_, beta1_, beta2_, beta3_, tau1_, tau2_;
public:
    term_structure_class_svensson(const double& beta0,
        const double& beta1,
        const double& beta2,
        const double& beta3,
        const double& tau1,
        const double& tau2);
    virtual double yield(const double& T) const;
};
```

Header file 22.2: Header file defining a term structure class wrapper for the Svensson model

```cpp
#include "fin_recipes.h"

term_structure_class_svensson::term_structure_class_svensson( const double& b0, const double& b1, const double& b2, const double& b3,
    const double& tau1, const double& tau2) {
    beta0_ = b0; beta1_ = b1; beta2_ = b2; beta3_ = b3; tau1_ = tau1; tau2_ = tau2;
};

double term_structure_class_svensson::yield(const double& t) const {
    if (t<=0.0) return beta0_;
    return term_structure_yield_svensson(t,beta0_,beta1_,beta2_,beta3_,tau1_,tau2_);
};
```

C++ Code 22.4: Defining a term structure class wrapper for the Svensson model
22.3 Cubic spline.

Cubic splines are well known for their good interpolation behaviour. The cubic spline parameterization was first used by McCulloch (1971) to estimate the nominal term structure. He later added taxes in McCulloch (1975). The cubic spline was also used by Litzenberger and Rolfo (1984). In this case the cubic spline is used to approximate the discount factor, not the yields.

\[
d(t) = 1 + b_1 t + c_1 t^2 + d_1 t^3 + \sum_{j=1}^{K} F_j (t - t_j)^3 1_{(t < t_j)}
\]

Here \(1_{(A)}\) is the indicator function for an event \(A\), and we have \(K\) knots.

To estimate this we need to find the \(3 + K\) parameters:

\[\{b_1, c_1, d_1, F_1, \ldots, F_K\}\]

If the spline knots are known, this is a simple linear regression. C++ Code 22.5 shows the calculation using this approximation.

```cpp
#include <cmath>
#include <vector>

using namespace std;

double term_structure_discount_factor_cubic_spline(const double& t,
                                                    const double& b1,
                                                    const double& c1,
                                                    const double& d1,
                                                    const vector<double>& f,
                                                    const vector<double>& knots){
    double d = 1.0 + b1*t + c1*(pow(t,2)) + d1*(pow(t,3));
    for (int i=0;i<knots.size();i++) {
        if (t >= knots[i]) { d += f[i] * (pow((t-knots[i]),3)); }
        else { break; }
    }
    return d;
}
```

C++ Code 22.5: Approximating a discount function using a cubic spline

Header File 22.3 and C++ Code 22.6 wraps this calculations into a term structure class.

```cpp
#include "fin_recipes.h"
#include <vector>

using namespace std;

class term_structure_class_cubic_spline : public term_structure_class {
private:
    double b_; double c_; double d_; 
    vector<double> f_; vector<double> knots_; 
public:
    term_structure_class_cubic_spline(const double& b, const double& c, const double& d,
                                      const vector<double>& f, const vector<double>& knots);
    virtual "term_structure_class_cubic_spline() ;
    virtual double d(const double& T) const;
};
```

Header file 22.3: Term structure class wrapping the cubic spline approximation
#include "fin_recipes.h"

term_structure_class::cubic_spline::

term_structure_class::cubic_spline ( const double & b, const double & c, const double & d,
const vector<double> & f, const vector<double> & knots ) {
    b_ = b; c_ = c; d_ = d; f_.clear(); knots_.clear();
    if (f_.size() != knots_.size()) { return; };
    for (int i = 0; i < f_.size(); ++i) {
        f_.push_back(f_[i]);
        knots_.push_back(knots_[i]);
    };
};

double term_structure_class::cubic_spline::d(const double & T) const {
    return term_structure_discount_factor_cubic_spline(T, b_, c_, d_, f_, knots_);
};

C++ Code 22.6: Term structure class wrapping the cubic spline approximation
Example

Using the parameters \( b = 0.1 \), \( c = 0.1 \), \( d = -0.1 \),

\[
\begin{bmatrix}
0.01 \\
0.01 \\
-0.01
\end{bmatrix}
\]

\( f = \)

\[
\begin{bmatrix}
2 \\
7 \\
12
\end{bmatrix}
\]

\( \text{knots} = \)

Find short rates and discount factors for 1 year, and the forward rate between 1 and 2 years.

\begin{verbatim}
C++ program:

cout << "Example term structure calculations using a cubic spline " << endl;
double b=0.1; double c=-0.1; double d=-0.1;
vector<double> f; f.push_back(0.01); f.push_back(0.01); f.push_back(-0.01);
vector<double> knots; knots.push_back(2); knots.push_back(7); knots.push_back(12);
cout << " direct calculation, discount factor (t=1) "
   << term_structure_discount_factor_cubic_spline(1,b,c,d,f,knots) << endl;
cout << " Using a term structure class " << endl;
term_structure_class_cubic_spline cs(b,c,d,f,knots);
cout << " yield (t=1) = " << cs.r(1) << endl;
cout << " discount factor (t=1) = " << cs.d(1) << endl;
cout << " forward (t1=1, t2=2) = " << cs.f(1,2) << endl;
\end{verbatim}

Output from C++ program:

Example term structure calculations using a cubic spline
direct calculation, discount factor (t=1) 1.1
Using a term structure class
yield (t=1) = -0.0953102
discount factor (t=1) = 1.1
forward (t1=1, t2=2) = 0.318454
22.4 Cox Ingersoll Ross.

The Cox et al. (1985) model is the best known example of a continuous time, general equilibrium model of the term structure. It is commonly used in academic work because it is a general equilibrium model that still is “simple enough” to let us find closed form expressions for derivative securities.

The short interest rate.

\[ dr(t) = \kappa(\theta - \tau(t))dt + \sigma \sqrt{\tau(t)}dW \]

The discount factor for a payment at time T.

\[ d(t, T) = A(t, T)e^{-B(t, T)\tau(t)} \]

where

\[ \gamma = \sqrt{(\kappa + \lambda)^2 + 2\sigma^2} \]

\[ A(t, T) = \left[ \frac{2\gamma e^{\frac{1}{2}(\kappa + \lambda + \gamma)(T-t)}}{(\gamma + \kappa + \lambda)(e^{\lambda(T-t)} - 1) + 2\gamma} \right]^{\frac{T-t}{\gamma^2}} \]

and

\[ B(t, T) = \frac{2e^{\gamma(T-t)} - 1}{(\gamma + \kappa + \lambda)(e^{\lambda(T-t)} - 1) + 2\gamma} \]

Five parameters: \( \tau \), the short term interest rate, \( \kappa \), the mean reversion parameter, \( \lambda \), the “market” risk parameter, \( \theta \) the long–run mean of the process and \( \sigma \), the variance rate of the process.

C++ Code 22.7: Calculation of the discount factor using the Cox et al. (1985) model

```cpp
#include <cmath>
using namespace std;

doctor term_structure_discount_factor_cir(const double& t,
    const double& r, 
    const double& kappa,
    const double& lambda,
    const double& theta,
    const double& sigma){

doctor sigma_sqr=pow(sigma,2);
doctor gamma = sqrt(pow((kappa+lambda),2)+2.0*sigma_sqr);
doctor denum = (gamma+kappa+lambda)*(exp(gamma*t)-1)+2*gamma;
doctor enum1= 2*gamma*exp(0.5*(kappa+lambda+gamma)*t);
doctor A = pow(enum1/denum,p);
doctor B = (2*(exp(gamma*t)-1))/denum;
doctor dfact=A*exp(-B*t);
return dfact;
};
```
#include "fin_recipes.h"

class term_structure_class_cir : public term_structure_class {
private:
    double r_;        // interest rate
    double kappa_;    // mean reversion parameter
    double lambda_;   // risk aversion
    double theta_;    // long run mean
    double sigma_;    // volatility
public:
    ~term_structure_class_cir();
    term_structure_class_cir(const double & r, const double & k, const double & l, const double & th, const double & sigma);
    virtual double d(const double & T) const;
};

Header file 22.4: Class definition, Cox et al. (1985) model, header file

#include "fin_recipes.h"

//term_structure_class_cir::`term_structure_class_cir();

term_structure_class_cir::term_structure_class_cir(const double & r, const double & k, const double & l, const double & th, const double & sigma) {
    r_=r; kappa_=k; lambda_=l; theta_=th; sigma_=sigma;
};

double term_structure_class_cir::d(const double & T) const{
    return term_structure_discount_factor_cir(T, r_, kappa_, lambda_, theta_, sigma_);
};

C++ Code 22.8: Class definition, Cox et al. (1985) model
Example

Parameters: \( r = 0.05, \kappa = 0.01, \sigma = 0.1, \theta = 0.08 \) and \( \lambda = 0.0 \). Use the CIR term structure model. Find short rate and discount factor for \( t = 1 \), and forward rate between years 1 and 2.

C++ program:

```cpp
cout << "Example calculations using the Cox Ingersoll Ross term structure model " << endl;
double r = 0.05; double kappa=0.01; double sigma=0.1; double theta=0.08; double lambda=0.0;
cout << " direct calculation, discount factor (t=1): "
    << term_structure_discount_factor_cir(1, r, kappa, lambda, theta, sigma) << endl;
cout << " using a class " << endl;
term_structure_class_cir cir(r, kappa, lambda, theta, sigma);
cout << " yield (t=1) = " << cir.r(1) << endl;
cout << " discount factor (t=1) = " << cir.d(1) << endl;
cout << " forward (t1=1, t2=2) = " << cir.f(1,2) << endl;
```

Output from C++ program:

Example calculations using the Cox Ingersoll Ross term structure model

direct calculation, discount factor (t=1): 0.951166

using a class

yield (t=1) = 0.0500668
discount factor (t=1) = 0.951166
forward (t1=1, t2=2) = 0.0498756
#include <cmath>
using namespace std;

double term_structure_discount_factor_vasicek(const double& time,
    const double& r,
    const double& a,
    const double& b,
    const double& sigma){
    double A,B;
    double sigma_sqr = sigma*sigma;
    double aa = a*a;
    if (a==0.0){
        B = time;
        A = exp(sigma_sqr*pow(time,3))/6.0;
    }
    else {
        B = (1.0 - exp(-a*time))/a;
        A = exp( ((B-time)*(aa*b-0.5*sigma_sqr))/aa +((sigma_sqr*B*B)/(4*a)));
    }
    double dfact = A*exp(-B*r);
    return dfact;
}

C++ Code 22.9: Calculating a discount factor using the Vasicek functional form

#include "fin_recipes.h"

class term_structure_class_vasicek : public term_structure_class {
private:
    double r_; double a_; double b_; double sigma_;
public:
    term_structure_class_vasicek(const double& r, const double& a, const double& b, const double& sigma);
    virtual double discount_factor(const double& T) const;
};

Header file 22.5: Class definition, Vasicek (1977) model
#include "fin_recipes.h"

term_structure_class vasicek::term_structure_class_vasicek(const double& r, const double& a,
                        const double& b, const double& sigma) {
    r_=r; a_=a; b_=b; sigma_=sigma;
};

double term_structure_class_vasicek::d(const double& T) const{
    return term_structure_discount_factor_vasicek(T,r_,a_,b_,sigma_);
};

C++ Code 22.10: Class definition, Vasicek (1977) model
Example

Parameters $\tau = 0.05$, $a = -0.1$, $b = 0.1$, $\sigma = 0.1$ Use the Vasicek term structure model. Find short rate and discount factor for $t = 1$, and forward rate between years 1 and 2.

C++ program:

```cpp
cout << "Example term structure calculation using the Vasicek term structure model" << endl;
double r=0.05; double a=-0.1; double b=0.1; double sigma=0.1;
cout << " direct calculation, discount factor (t=1): "
<< term_structure_discount_factor_vasicek(1, r, a, b, sigma) << endl;
term_structure_class_vasicek vc(r,a,b,sigma);
cout << " using a term structure class " << endl;
cout << " yield (t=1) = " << vc.r(1) << endl;
cout << " discount factor (t=1) = " << vc.d(1) << endl;
cout << " forward rate (t1=1, t2=2) = " << vc.f(1,2) << endl;
```

Output from C++ program:

Example term structure calculation using the Vasicek term structure model
direct calculation, discount factor (t=1): 0.955408
using a term structure class
yield (t=1) = 0.0456168
discount factor (t=1) = 0.955408
forward rate (t1=1, t2=2) = 0.0281476

22.6  Readings

The methods in this chapter I first studied in my dissertation at Carnegie Mellon University in 1992, which lead to the paper published as Green and Ødegaard (1997). A textbook treatment of estimation and fitting of term structure models can be found in (Martinelli, Priaulet, and Priaulet, 2003, Ch 4)
Pricing bond options with the Black Scholes model, or its binomial approximation, as done in chapter 20, does not always get it right. For example, it ignores the fact that at the maturity of the bond, the bond volatility is zero. The bond volatility decreases as one gets closer to the bond maturity. This behaviour is not captured by the assumptions underlying the Black Scholes assumption. We therefore look at more complicated term structure models, the unifying theme of which is that they are built by building trees of the interest rate.

23.1 The Rendleman and Bartter model

The Rendleman and Bartter approach to valuation of interest rate contingent claims (see Rendleman and Bartter (1979) and Rendleman and Bartter (1980)) is a particular simple one. Essentially, it is to apply the same binomial approach that is used to approximate options in the Black Scholes world, but the random variable is now the interest rate. This has implications for multiperiod discounting: Taking the present value is now a matter of choosing the correct sequence of spot rates, and it may be necessary to keep track of the whole “tree” of interest rates. Such a tree can then be used to price various fixed income securities. In the next chapter we illustrate this more generally, here we show a direct implementation of the original model. C++ Code 23.1 implements the original algorithm for a call option on a (long maturity) zero coupon bond.
#include <cmath>
#include <algorithm>
#include <vector>

using namespace std;

double bond_option_price_call_zero_american_rendleman_bartter(const double& X,
    const double& option_maturity,
    const double& S,
    const double& M, // term structure parameters
    const double& interest, // current short interest rate
    const double& bond_maturity, // time to maturity for underlying bond
    const double& maturity_payment,
    const int& no_steps) {

    double delta_t = bond_maturity/no_steps;

    double u=exp(S*sqrt(delta_t));
    double d=1/u;
    double p_up = (exp(M*delta_t)-d)/(u-d);
    double p_down = 1.0-p_up;

    vector<double> r(no_steps+1);
    r[0]=interest*pow(d,no_steps);
    double uu=u*u;
    for (int i=1;i<no_steps+1;i++)
        r[i]=r[i-1]*uu;
    vector<double> P(no_steps+1);
    P[0]=maturity_payment;
    for (int i=0; i<no_steps+1; i++)
        P[i]=exp(-r[i]*delta_t)*(p_down*P[i]+p_up*P[i+1]);

    vector<double> C(no_call_steps+1);
    for (int i=0; i<no_call_steps+1; i++)
        C[i]=max(0.0,P[i]-X);
    for (int curr_step=no_call_steps; curr_step>=0; curr_step--)
        for (int i=0; i<curr_step+1; i++)
            r[i]=r[i]*u;
        P[i]=exp(-r[i]*delta_t)*(p_down*P[i]+p_up*P[i+1]);
        C[i]=max(P[i]-X, exp(-r[i]*delta_t)*(p_up*C[i+1]+p_down*C[i]));
    return C[0];
}

C++ Code 23.1: RB binomial model for European call on zero coupon bond
**Example**

Parameters: $K = 950$, $S = 0.15$ and $M = 0.05$. The interest rate is 10%, The option matures in 4 years, the bond matures in year 5, with a bond maturity payment of 1000. Price the option on the zero coupon bond using a Randleman–Bartter approximation with 100 steps.

```cpp
double K=950; double S=0.15; double M=0.05; double interest=0.10;
double option_maturity=4; double bond_maturity=5; double bond_maturity_payment=1000;
int no_steps=100;
cout << " Rendleman Bartter price of option on zero coupon bond: ";
cout << bond_option_price_call_zero_american_rendleman_bartter( K, option_maturity, S, M,
                   interest, bond_maturity,
                   bond_maturity_payment, no_steps);
```

**Output from C++ program:**

Rendleman Bartter price of option on zero coupon bond: 0.00713661

---

**23.2 Readings**

General references include Sundaresan (2001).
Rendleman and Bartter (1979) and Rendleman and Bartter (1980) are the original references for building standard binomial interest rate trees.
Chapter 24

Interest rate trees

Contents

24.1 The movement of interest rates ......................................................... 214
24.2 Discount factors ............................................................................. 216
24.3 Pricing bonds ................................................................................ 216
24.4 Callable bond ................................................................................ 218
24.5 Readings ....................................................................................... 220

In this chapter we show a way of building interest rate trees and apply it to the pricing of various fixed
income securities. The first need in such a procedure is to specify the future evolution of interest rates.

24.1 The movement of interest rates

We will assume that interest rates follow a Geometric Brownian Motion process

\[ dr = \mu r dt + \sigma r dZ \]

This is the same assumption which was used for stock prices in the Black Scholes case. This leads to
the same binomial approximation, for one period:

or several periods:
When using this approach it turns out to be useful to keep the whole tree around, we therefore separate the building of the tree in one routine, as shown in C++ Code 24.1.

```cpp
#include <vector>
#include <cmath>
using namespace std;

vector<vector<double>> interest_rate_trees_gbm_build(const double &r0,
    const double &u,
    const double &d,
    const int &n){

    vector<vector<double>> tree;
    vector<double> r(1); r[0]=r0;
    tree.push_back(r);
    for (int i=1;i<n;++i) {
        double rtop=r[tree.size()-1]*u;
        for (int j=0;j<i;++j){
            r[j] = d*r[j];
        }
        r.push_back(rtop);
        tree.push_back(r);
    }
    return tree;
}
```

C++ Code 24.1: Building interest rate tree

```matlab
function tree = interest_rate_trees_gbm_build(r0,u,d,n)
    r=[r0];
    tree=[r];
    rtop=r0;
    for i=1:n
        rtop = u * rtop;
        r = [rtop,d*r];
        tree=[ zeros(1,i),tree,r ];
    endfor
endfunction
```

Matlab Code 24.1: Building interest rate tree

**Example**

Parameters: \( r_0 = 10\% \), \( u = 1.02 \), \( d = 0.99 \). Build a 2 period interest rate tree.
C++ program:

```cpp
vector<vector<double>> tree = interest_rate_trees_gbm_build(0.1, 1.02, 0.99, 3);
cout << " Interest rate tree: " << endl;
cout << " Time 0: " << tree[0][0] << endl;
cout << " Time 1: " << * tree[1][0] << " " << * tree[1][1] << endl;
```

Output from C++ program:

```
Interest rate tree:
Time 0: 0.1
Time 1: 0.099 0.102
Time 2: 0.09801 0.10098 0.10404
```

24.2 Discount factors

We want to price bonds and other fixed income securities, which contains sets of future cash flows. Instead of interest rates we need prices, not interest rates, i.e. discount factors. The interest rate tree therefore needs to be used to generate discount factors, which needs to be specified at every point of the tree, and for all relevant maturities. Let $d(t; T)$ be the discount factor at time $t$ for a time $T$ payment. If we at time 0 want to price cash flows at time 2, we need the following set of discount factors

$$d(0; 2)$$

$$d_u(1; 2)$$

$$d_d(1; 2)$$

In constructing trees of discount we need one additional piece of information, $q$, which are used as follows:

$$d(0, 2) = e^{-r_0} (q d_u(1, 2) + (1 + q) d_d(1, 2))$$

The parameter $q$ serves the same purpose as the state price probability, but it is found differently.

Exercise 24.1.

Given the prices of two discount bonds, with maturities 1 and 2, how would you back out $q$?

Exercise 24.2.

Suppose you have $q$ and the tree of short interest rates. How would you calculate the $t$-period spot rate?

24.3 Pricing bonds

Pricing straight bonds is then just a matter of the usual recursive valuation

Example

Parameters: $r_0 = 10\%$, $u = 1.02$, $d = 0.99$. Let $q = 0.5$. Determine the price of a 3 period coupon bond with coupon of 10 and face value of 100.
#include <vector>
#include <cmath>
using namespace std;

double interest_rate_trees_gbm_value_of_cashflows(const vector<double>& cflow,
                                                  const vector<vector<double>>& r_tree,
                                                  const double& q){
    int n = int(cflow.size());
    vector<vector<double>> values(n);
    vector<double> value(n);
    for (int i=0;i<n;++i){
        value[i]=cflow[n-1];
    }
    values[n-1]=value;
    for (int t=n-1;t>0;--t){
        vector<double> value(t,0.0);
        for (int i=0;i<t;++i){
            value[i]=cflow[t-1]+exp(-r_tree[t-1][i])*(q*value[t][i]+(1-q)*value[t][i+1]);
        }
        values[t-1]=value;
    }
    return values[0][0];
}

C++ Code 24.2: Valuing cash flows

C++ program:

double r0=0.1;
double u=1.02; double d=0.99;
int n=3;
double q=0.5;
vector<vector<double>> tree = interest_rate_trees_gbm_build(r0,u,d,n);
vector<double> cashflows;
cashflows.push_back(0); cashflows.push_back(10); cashflows.push_back(10); cashflows.push_back(10); cashflows.push_back(110);
cout << "Bond price B = " << interest_rate_trees_gbm_value_of_cashflows(cashflows,tree,q);

Output from C++ program:
Bond price B = 98.5997

Exercise 24.3.
The example just presented assumes cash flows dates matches the dates of interest rate changes. How would you modify the code to allow for differences in timing of interest rate changes and cashflows. Or can you do this alternatively, by adjusting the inputs to the routine?
24.4 Callable bond

A slightly more involved example is a callable bond.

```cpp
#include <vector>
#include <cmath>
using namespace std;

double interest_rate_trees_gbm_value_of Callable bond(const vector<double>& cflows,
    const vector<vector<double>>& r_tree,
    const double& q,
    const int& first_call_time,
    const double& call_price){
    int n = int(cflows.size());
    vector<vector<double>> values(n);
    vector<double> value(n);
    for (int i=0; i<n; i++) { value[i]=cflows[n-1]; }
    value[n-1]=value;
    for (int t=n-1; t>0; --t){
        vector<double> value(t, 0.0);
        for (int i=0; i<t; ++i){
            value[i]=cflows[t-1]+exp(-r_tree[t-1][i]*q*values[t][i]+(1-q)*values[t][i+1]);
            if (t==first_call_time){ value[i]=min(value[i],call_price); }
        }
        values[t-1]=value;
    }
    return values[0][0];
}
```

C++ Code 24.3: Valuing callable bond

Example

Construct a short rate lattice for periods (years) 0 through 9 with an initial rate of \( r_0 = 6\% \) and with successive rates determined by a multiplicative factors \( u = 1.2 \) or \( d = 0.9 \). Assign \( q = 0.5 \).

1. Using this lattice, find the value of a 10-year 6\% bond.

2. Suppose this bond can be called by the issuing party at any time after 5 years. (When the bond is called, the face value plus the currently due coupon are paid at that time and the bond is canceled.) What is the fair value of this bond?

The interest rate lattice:

<table>
<thead>
<tr>
<th>step</th>
<th>nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>31.0</td>
</tr>
<tr>
<td></td>
<td>25.8</td>
</tr>
<tr>
<td></td>
<td>23.2</td>
</tr>
<tr>
<td>1</td>
<td>21.5</td>
</tr>
<tr>
<td></td>
<td>19.3</td>
</tr>
<tr>
<td></td>
<td>17.4</td>
</tr>
<tr>
<td>2</td>
<td>17.9</td>
</tr>
<tr>
<td></td>
<td>16.1</td>
</tr>
<tr>
<td></td>
<td>14.5</td>
</tr>
<tr>
<td></td>
<td>13.1</td>
</tr>
<tr>
<td>3</td>
<td>14.9</td>
</tr>
<tr>
<td></td>
<td>13.4</td>
</tr>
<tr>
<td></td>
<td>12.1</td>
</tr>
<tr>
<td></td>
<td>10.9</td>
</tr>
<tr>
<td></td>
<td>9.8</td>
</tr>
<tr>
<td>4</td>
<td>12.4</td>
</tr>
<tr>
<td></td>
<td>11.2</td>
</tr>
<tr>
<td></td>
<td>10.1</td>
</tr>
<tr>
<td></td>
<td>9.1</td>
</tr>
<tr>
<td></td>
<td>8.2</td>
</tr>
<tr>
<td>5</td>
<td>10.4</td>
</tr>
<tr>
<td></td>
<td>9.3</td>
</tr>
<tr>
<td></td>
<td>8.4</td>
</tr>
<tr>
<td></td>
<td>7.6</td>
</tr>
<tr>
<td></td>
<td>6.8</td>
</tr>
<tr>
<td></td>
<td>6.1</td>
</tr>
<tr>
<td>6</td>
<td>8.6</td>
</tr>
<tr>
<td></td>
<td>7.8</td>
</tr>
<tr>
<td></td>
<td>7.0</td>
</tr>
<tr>
<td></td>
<td>6.3</td>
</tr>
<tr>
<td></td>
<td>5.7</td>
</tr>
<tr>
<td>7</td>
<td>5.1</td>
</tr>
<tr>
<td></td>
<td>4.6</td>
</tr>
<tr>
<td></td>
<td>4.1</td>
</tr>
<tr>
<td>8</td>
<td>7.2</td>
</tr>
<tr>
<td></td>
<td>6.5</td>
</tr>
<tr>
<td></td>
<td>5.8</td>
</tr>
<tr>
<td></td>
<td>5.2</td>
</tr>
<tr>
<td></td>
<td>4.7</td>
</tr>
<tr>
<td>9</td>
<td>4.3</td>
</tr>
<tr>
<td></td>
<td>3.8</td>
</tr>
<tr>
<td></td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>3.1</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
</tr>
<tr>
<td>10</td>
<td>6.0</td>
</tr>
<tr>
<td></td>
<td>5.4</td>
</tr>
<tr>
<td></td>
<td>4.9</td>
</tr>
<tr>
<td></td>
<td>4.4</td>
</tr>
<tr>
<td></td>
<td>3.9</td>
</tr>
<tr>
<td></td>
<td>3.5</td>
</tr>
<tr>
<td></td>
<td>3.2</td>
</tr>
<tr>
<td></td>
<td>2.9</td>
</tr>
<tr>
<td></td>
<td>2.6</td>
</tr>
<tr>
<td></td>
<td>2.3</td>
</tr>
</tbody>
</table>

1. Valuing the noncallable bond

The cash flow lattice
The value lattice:

<table>
<thead>
<tr>
<th>step</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>nodes:</td>
<td>106.00</td>
<td>83.78</td>
<td>73.14</td>
<td>68.67</td>
<td>67.84</td>
<td>69.38</td>
<td>72.59</td>
<td>77.07</td>
<td>82.53</td>
<td>88.70</td>
<td>99.00</td>
</tr>
</tbody>
</table>

The value of the bond is $B = 89.90$.

2. The callable will be called when the value of the bond is above par, because the issuing company can issue a bond at a lower interest rate.

The following set of values is calculated

<table>
<thead>
<tr>
<th>step</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>nodes:</td>
<td>106.00</td>
<td>83.78</td>
<td>73.14</td>
<td>68.67</td>
<td>67.84</td>
<td>69.38</td>
<td>72.59</td>
<td>77.07</td>
<td>82.53</td>
<td>88.70</td>
<td>99.00</td>
</tr>
</tbody>
</table>

The value of the bond is $B = 89.35$. 

219
**Exercise 24.4.**

How would you price a call option on a coupon bond in this setting?

### 24.5 Readings

General references include Sundaresan (2001).

Rendleman and Bartter (1979) and Rendleman and Bartter (1980) are the original references for building standard binomial interest rate trees.
Chapter 25

Building term structure trees using the Ho and Lee (1986) approach

25.1 Intro

In this section we build interest rate trees following the original paper of Ho and Lee (1986). We will follow the analysis in the paper, and use it to illustrate how you can build trees of more complex term structures than the simple binomial trees of the interest rate. The selling point of the original paper was that one could fit an initial term structure and then specify an evolution of the term structure consistent with this initial term structure.

25.2 Building trees of term structures

25.3 Ho Lee term structure class
#include "fin_recipes.h"

class term_structure_class_ho_lee : public term_structure_class {
private:
  term_structure_class* initial_term_;  
  int n_;  
  int i_;  
  double delta_;  
  double pi_;  
public:
  term_structure_class_ho_lee(term_structure_class* fitted_term,
       const int & n,
       const int & i,
       const double& lambda,
       const double& pi);  
  double d(const double& T) const;  
};  
vector<vector<term_structure_class_ho_lee>>
term_structure_ho_lee_build_term_structure_tree(term_structure_class* initial,
       const int& no_steps,
       const double& delta,
       const double& pi);  

double price_european_call_option_on_bond_using_ho_lee(term_structure_class* initial,
       const double& delta,
       const double& pi,
       const vector<double>& underlying_bond_cflow_times,
       const vector<double>& underlying_bond_cflows,
       const double& K,
       const double& option_time_to_maturity);

Header file 25.1: Term structure class for Ho-Lee

C++ Code 25.1: Term structure class for Ho-Lee

#include "fin_recipes.h"

term_structure_class_ho_lee::term_structure_class_ho_lee(term_structure_class* fitted_term,
       const int & n,
       const int & i,
       const double& delta,
       const double& pi){
  initial_term_=fitted_term;
  n_=n;
  i_=i;
  delta_=delta;
  pi_=pi;  
};
inline double hT(const double & T, const double & delta, const double & pi){
    return (1.0/(pi+(1-pi)*pow(delta,T)));
};

double term_structure_class_ho_lee::d(const double & T) const{
    double d=(*initial_term_).d(T+n_)/(*initial_term_).d(n_);
    for (int j=1;j<n_+1;++j){
        d *= hT(T+(n_-j).delta_..pi) / hT(n_-j.delta_..pi) ;
    }
    d *= hT(T.delta_..pi)*pow(delta_,T*(n_-i_));
    return d;
};

C++ Code 25.2: Term structure class for Ho-Lee, calculation of discount function

#include "fin_recipes.h"
vector< vector<term_structure_class_ho_lee> >
term_structure_ho_lee_build_term_structure_tree(term_structure_class* initial,
    const int & no_steps,
    const double & delta,
    const double & pi){
    vector< vector<term_structure_class_ho_lee> > hl_tree;
    for (int t=0;t<5;++t){
        hl_tree.push_back(vector<term_structure_class_ho_lee>());
        for (int j=0;j<=t;++j){
            term_structure_class_ho_lee hl(initial,t,j,delta,pi);
            hl_tree[t].push_back(hl);
        }
    }
    return hl_tree;
};

C++ Code 25.3: Building a term structure tree
25.4 Pricing things

We now have access to what we need to do pricing through the recursive relationship

\[ C(n, i) = [\pi C(n + 1, i + 1) + (1 - \pi)C(n + i, i)] P_i^{(n)}(1) \]

where \( C(n, i) \) is the value of a security at time \( n \) at node \( i \).

What we are pricing are typically state and time-contingent claims to cash flows. Let us illustrate pricing of an (European) call option on some underlying bond. Suppose this bond is risk free. Its cash flows at each future date does not depend on the state, but the timing of cash flows changes as you move in the tree. It is therefore necessary to some way figure out at date \( t \): What are the future cash flows when you want to price the underlying bond at that date. We build a small class that contains this information, and use it, together with the term structures in the individual nodes, to find the bond price at each node. The value of the bond at the different nodes change because the term structure you use for discounting is changing. This bond price is then used to find the option value at each node.
#include "fin_recipes.h"

class contingent_cash_flows{
  public:
    vector<double> times;
    vector<double> cash_flows;
  
    time_contingent_cash_flows(const vector<double>& in_times, const vector<double>& in_cflows){
      times=in_times; cash_flows=in_cflows;
    }
    
    int no_cflows(){ return int(times.size()); }

    vector<time_contingent_cash_flows> build_time_series_of_bond_time_contingent_cash_flows(const vector<double>& initial_times, const vector<double>& initial_cflows){
      vector<time_contingent_cash_flows> vec_cf;
      vector<double> times = initial_times;
      vector<double> cflows = initial_cflows;
      
      while (times.size()>0){
        vec_cf.push_back(time_contingent_cash_flows(times,cflows));
        
        vector<double> tmp_times;
        vector<double> tmp_cflows;
        for (int i=0; i<times.size(); i++){
          if (times[i]-1.0>=0.0) {
            tmp_times.push_back(times[i-1]);
            tmp_cflows.push_back(cflows[i]);
          }
        }
        times = tmp_times; cflows = tmp_cflows;
      }
      return vec_cf;
    }

    double price_european_call_option_on_bond_using_ho_lee(term_structure_class* initial, const double& delta, const double& pi, const vector<double>& underlying_bond_cflow_times, const vector<double>& underlying_bond_cflows, const double& K, const double& time_to_maturity){
      int T = int(time_to_maturity+0.0001);
      vector<vector<term_structure_class>> hl_tree;
      for (int i=0;i<T;i++){
        hl_tree[i].push_back(initial);
      }
      
      vector<double> values(T+1);
      for (int i=0;i<T;i++){
        values[i]=max(0.0, bonds_price(vec_cf[i].times, vec_cf[i].cash_flows, hl_tree[i][i]) - K);
      }
      
      for (int i=0;i<T;i++){
        values[i]=max(0.0, bonds_price(vec_cf[i].times, vec_cf[i].cash_flows, hl_tree[i][i]) - K);
      }
      return values[T];
  }

  
  double price_european_call_option_on_bond(term_structure_class* initial, const double& delta, const double& pi, const vector<double>& bond_cflow_times, const vector<double>& bond_cflows, const double& K, const double& time_to_maturity){
    
    double price_european_call_option_on_bond_using_ho_lee(initial, delta, pi, bond_cflow_times, bond_cflows, K, time_to_maturity);
  }

  };

}
Example
You are pricing options on a 5 year zero coupon risk free bond. The options are European calls with a time
to maturity of 3 years.
You will price the options using a Ho-Lee approach with parameters $\pi = 0.5$ and $\delta = 0.98$.
Price the option using two different assumptions about the current term structure:

1. The term structure is flat with an interest rate of 10% (continously compounded).

2. The current term structure has been estimated using a Nelson Siegel parameterization with parameters
   $\beta_0 = 0.09$, $\beta_1 = 0.01$, $\beta_2 = 0.01$ and $\lambda = 5.0$.

C++ program:

```cpp
double delta=0.98;
double pi=0.5;
double r=0.1;
term_structure_class* initial=new term_structure_class_flat(r);
vector<double> times; times.push_back(5.0);
vector<double> cflows; cflows.push_back(100);
double K=80;
double time_to_maturity=3;
cout << " Flat term structure " << endl;
cout << " c= " << price_european_call_option_on_bond_using_ho_lee(initial, delta, pi, times, cflows, K, time_to_maturity) << endl;
delete (initial);
double beta0=0.09; double beta1=0.01; double beta2=0.01; double lambda=5.0;
initial = new term_structure_class_nelson_siegel(beta0, beta1, beta2, lambda);
cout << " Nelson Siegel term structure " << endl;
cout << " c= " << price_european_call_option_on_bond_using_ho_lee(initial, delta, pi, times, cflows, K, time_to_maturity) << endl;
cout << endl;
```

Output from C++ program:

Flat term structure
c = 6.46323
Nelson Siegel term structure
c = 6.33238

Exercise 25.1.
What changes do you need to make to C++ Code 25.4 to price an American call option instead of an European
option?

Exercise 25.2.
If you for example want to price a bond, you need to keep track of intermediate payments of coupon.
Implement such a procedure. To see that it is correct us it to price a (straight) bond and then compare the
value calculated in the tree with the value using the current term structure. Then modify the code to build
in a callable bond feature. What additional informaiton must be kept track of?

Exercise 25.3.
In the Ho and Lee (1986) paper there is a typo in equation (22). Can you see what it is?

25.5 References
The discussion in this chapter follows closely the original paper Ho and Lee (1986)
Chapter 26

Term Structure Derivatives

Contents

26.1 Vasicek bond option pricing ........................................ 227

26.1 Vasicek bond option pricing

If the term structure model is Vasicek’s model there is a solution for the price of an option on a zero coupon bond, due to Jamshidan (1989).

Under Vasicek’s model the process for the short rate is assumed to follow.

\[
dr = a(b - r)dt + \sigma dZ
\]

where \(a\), \(b\) and \(\sigma\) are constants. We have seen earlier how to calculate the discount factor in this case. We now want to consider an European Call option in this setting.

Let \(P(t, s)\) be the time \(t\) price of a zero coupon bond with a payment of $1 at time \(s\) (the discount factor). The price at time \(t\) of a European call option maturing at time \(T\) on on a discount bond maturing at time \(s\) is (See Jamshidan (1989) and Hull (1993))

\[
P(t, s)N(h) - X P(t, T)N(h - \sigma_P)
\]

where

\[
h = \frac{1}{\sigma_P} \ln \frac{P(t, s)}{P(t, T)X} + \frac{1}{2} \sigma_P
\]

\[
\sigma_P = v(t, T)B(T, s)
\]

\[
B(t, T) = \frac{1 - e^{-a(T-t)}}{a}
\]

\[
v(t, T)^2 = \frac{\sigma^2 (1 - e^{-a(T-t)})}{2a}
\]

In the case of \(a = 0\),

\[
v(t, T) = \sigma \sqrt{T - t}
\]

\[
\sigma_P = \sigma (s - T) \sqrt{T - t}
\]

Example

Parameters: \(a = 0.1, b = 0.1, \sigma = 0.02, r = 0.05, X = 0.9\). Price a Vasicek call on a zero.


```cpp
#include "normdist.h"
#include "fin_recipes.h"
#include <cmath>

using namespace std;

double bond_option_price_call_zero_vasicek(const double& X, // exercise price
    const double& r, // current interest rate
    const double& option_time_to_maturity,
    const double& bond_time_to_maturity,
    const double& a, // parameters
    const double& b,
    const double& sigma){

double T_t = option_time_to_maturity;
double s_t = bond_time_to_maturity;
double T_s = s_t-T_t;
double v_t_T;
double sigma_P;
if (a==0.0) {
v_t_T = sigma * sqrt ( T_t ) ;
sigma_P = sigma*T_s*sqrt(T_t);
}
else {
v_t_T = sqrt ( sigma*sigma*(1-exp(-2*a*T_t))/(2*a) );
double B_T_s = (1-exp(-a*T_s))/a;
sigma_P = v_t_T*B_T_s;
}

double h = (1.0/sigma_P) * log (term_structure_discount_factor_vasicek(s_t,r,a,b,sigma)/
    (term_structure_discount_factor_vasicek(T_t,r,a,b,sigma)*X) ) + sigma_P/2.0;
double c = term_structure_discount_factor_vasicek(s_t,r,a,b,sigma)*N(h) -X*term_structure_discount_factor_vasicek(T_t,r,a,b,sigma)*N(h-sigma_P);
return c;
};
```

**C++ Code 26.1:** Bond option pricing using the Vasicek model

C++ program:
```cpp
double a = 0.1; double b = 0.1; double sigma = 0.02; double r = 0.05; double X=0.9;
cout << " Vasicek call option price "
    << bond_option_price_call_zero_vasicek(X,r,1.5,a,b,sigma) << endl;
```

Output from C++ program:
```
Vasicek call option price 0.000226833
```
Chapter 27

Date (and time) revisited - the BOOST libraries

In the introduction we used the construction of a date class as an example of building a nontrivial (and useful) class. However, even if we follow the hints in the exercises, and use the operating system utilities for dealing with dates, at some point such a “homespun” date class needs to be replaced with something more reliable.

The date class that is most likely to spring to the programmers’ minds these days is one provided by the Boost C++ library. Let us therefore use this opportunity to introduce the Boost libraries, and show some examples of date usage. The Boost libraries are collections of C++ libraries, many of which may at some point become part of the C++ standard.¹ One of the most useful of the Boost libraries are complete libraries for date (gregorian) and date and time (date_time).

We are not going to go into detail here, we limit ourselves to showing some examples of usage:

¹Ten Boost libraries actually became part of the latest C++ standard (C++-11). If you have a recent C++ implementation, the Boost libraries are typically included.
C++ program:

```cpp
#include "boost/date_time/posix_time/posix_time.hpp"
#include <iostream>
using namespace std;
using namespace boost::gregorian;
using namespace boost::posix_time;

void ex_date_time(){
    boost::gregorian::date d(2014,4,1); // creating a date
    cout << " date: " << d << endl;

    boost::posix_time::ptime dt;
    stringstream ss("2014-mar-01 10:01:01.010000");
    ss >> dt;
    cout << " date time just created: " << dt << endl;

    ptime now = second_clock::local_time(); //get the current time from the clock
    date today = now.date(); //Get the date part out of the time
    cout << " today: " << today << endl;
    date tommorrow = today + days(1); // add one date
    ptime tommorrow_start(tommorrow); //midnight
    time_duration remaining = tommorrow_start - now;
    std::cout << "Time left till midnight: " << to_simple_string(remaining) << std::endl;
}
```

Output from C++ program:

```
date: 2014-Apr-01
date time just created: 2014-Mar-01 10:01:01.010000
today: 2014-Jun-30
Time left till midnight: 10:19:33
```

### 27.1 References

The Boost homepage is at [http://www.boost.org/](http://www.boost.org/)

230
Appendix A

Normal Distribution approximations.

We will in general not go into detail about more standard numerical problems not connected to finance, there are a number of well known sources for such, but we show the example of calculations involving the normal distribution.

A.1 The normal distribution function

The normal distribution function

\[ n(x) = e^{-\frac{x^2}{2}} \]

is calculated as

```c++
#include <cmath>  // c library of math functions
using namespace std;  // which is part of the standard namespace

// most C compilers define PI, but just in case it doesn't
#ifndef PI
#define PI 3.141592653589793238462643
#endif

double n(const double & x) {  // normal distribution function
  return (1.0/sqrt(2.0*PI))*exp(-0.5*x*x);
};
```

C++ Code A.1: The normal distribution function
A.2 The cumulative normal distribution

The solution of a large number of option pricing formulas are written in terms of the cumulative normal distribution. For a random variable \( x \) the cumulative probability is the probability that the outcome is lower than a given value \( z \). To calculate the probability that a normally distributed random variable with mean 0 and unit variance is less than \( z \), \( N(z) \), one have to evaluate the integral

\[
\text{Prob}(x \leq z) = N(z) = \int_{-\infty}^{z} n(x) \, dx = \int_{-\infty}^{z} e^{-\frac{x^2}{2}} \, dx
\]

There is no explicit closed form solution for calculation of this integral, but a large number of well known approximations exists. Abramowitz and Stegun (1964) is a good source for these approximations. The following is probably the most used such approximation, it being pretty accurate and relatively fast. The arguments to the function are assumed normalized to a (0,1) distribution.

```cpp
#include <cmath> // math functions.
using namespace std;

double N(const double& z) {
    if (z > 6.0) { return 1.0; }; // this guards against overflow
    if (z < -6.0) { return 0.0; };

double b1 = 0.31938153;
double b2 = 0.356563782;
double b3 = 1.781477937;
double b4 = -1.821255978;
double b5 = 1.330274429;
double p = 0.2316419;
double c2 = 0.3989423;
double a = fabs(z);
double t = 1.0/(1.0+a*p);
double b = c2*exp((-z)*(z/2.0));
double n = (((b5*t+b4)*t+b3)*t+b2)*t+b1)*t;
    n = 1.0-b*n;
    if (z < 0.0) n = 1.0 - n;
    return n;
};
```

C++ Code A.2: The cumulative normal

A.3 Multivariate normal

The normal distribution is also defined for several random variables. We then characterise the vector of random variables

\[
X = \begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{bmatrix}
\]

A probability statement about this vector is a joint statement about all elements of the vector.
A.4 Calculating cumulative bivariate normal probabilities

The most used multivariate normal calculation is the bivariate case, where we let \( x \) and \( y \) be bivariate normally distributed, each with mean 0 and variance 1, and assume the two variables have correlation of \( \rho \). By the definition of correlation \( \rho \in [-1, 1] \). The cumulative probability distribution

\[
P(x < a, y < b) = N(a, b, \rho) = \int_{-\infty}^{a} \int_{-\infty}^{b} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left( -\frac{1}{2} \frac{x^2 - 2\rho xy + y^2}{1 - \rho^2} \right) dx dy
\]

There are several approximations to this integral. We pick one such, discussed in (Hull, 1993, Ch 10), shown in C++ Code A.3.

If one has more than two correlated variables, the calculation of cumulative probabilities is a nontrivial problem. One common method involves Monte Carlo estimation of the definite integral. We will consider this, but then it is necessary to first consider simulation of random normal variables.

Example
Calculate \( N(0) \) and \( N(0,0,0) \)

<table>
<thead>
<tr>
<th>C++ program:</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>cout &lt;&lt; &quot;N(0) = &quot; &lt;&lt; N(0) &lt;&lt; endl;</code></td>
</tr>
<tr>
<td><code>cout &lt;&lt; &quot;N(0,0,0) = &quot; &lt;&lt; N(0,0,0) &lt;&lt; endl;</code></td>
</tr>
</tbody>
</table>

Output from C++ program:

\[
N(0) = 0.5  \\
N(0,0,0) = 0.25
\]
#include <cmath>  // include the standard library mathematics functions
#include <iostream> // which are in the standard namespace

double N(const double&);  // define the univariate cumulative normal distribution as a separate function

ifndef PI
const double PI=3.141592653589793238462643;
endif

inline double f(const double& x, const double& y, const double& aprime, const double& bprime, const double& rho) {
    double r = aprime*(2*x*aprime) + bprime*(2*y*bprime) + 2*rho*(x*aprime)*(y*bprime);
    return exp(r);
};

inline double sgn(const double& x) {  // sign function
    if (x>=0.0) return 1.0;
    return -1.0;
};

double N(const double& a, const double& b, const double& rho) {
    if ( (a<=0.0) && (b<=0.0) && (rho<=0.0) ) {
        double aprime = a/sqrt(2.0*(1.0-rho*rho));
        double bprime = b/sqrt(2.0*(1.0-rho*rho));
        double A[4]={0.3253030, 0.4211071, 0.1334425, 0.006374323};
        double B[4]={0.1337764, 0.6243247, 1.3425378, 2.2626645 };  
        double sum = 0;
        for (int i=0; i<4; i++) {
            for (int j=0; j<4; j++) {
                sum += A[i]*A[j]*f(B[i],B[j],aprime,bprime,rho);
            }
        }
        sum = sum * (sqrt(1.0-rho*rho)/PI);
        return sum;
    } else if ( (a*b*rho<=0.0) ) {
        if ( (a<=0.0) && (b>=0.0) && (rho>=0.0) ) {
            return N(a) - N(a, b, -rho);
        } else if ( (a>=0.0) && (b<=0.0) && (rho>=0.0) ) {
            return N(b) - N(-a, b, -rho);
        } else if ( (a>=0.0) && (b>=0.0) && (rho<=0.0) ) {
            return N(a) + N(b) - 1.0 + N(-a, -b, rho);
        };
    } else if ( (a*b*rho>=0.0) ) {
        double denum = sqrt(a*a - 2*rho*a*b + b*b);
        double rho1 = ((rho*a - b) * sgn(a))/denum;
        double rho2 = ((rho*b - a) * sgn(b))/denum;
        double delta=(1.0-sgn(a)*sgn(b))/4.0;
        return N(a,0.0,rho1) + N(b,0.0,rho2) - delta;
    } else {
        cout << " unknown " << endl;
    }
    return -99.9;  // should never get here, alternatively throw exception
};

C++ Code A.3: Approximation to the cumulative bivariate normal
A.5 Simulating random normal numbers

Generation of random numbers is a large topic and is treated at length in such sources as Knuth (1997). The generated numbers can never be truly random, only “pseudo”-random, they will be generated according to some reproducible algorithm and after a (large) number of random number generations the sequence will start repeating itself. The number of iterations before replication starts is a measure of the quality of a random number generator. For anybody requiring high-quality random number generators the `rand()` function provided by the standard C++ library should be avoided, but for not getting into involved discussion of random number generations we use this function as a basis for the generation of uniformly distributed numbers in the interval \([0, 1]\), as shown in C++ Code A.4.

```cpp
#include <cstdlib>
using namespace std;

double random_uniform_0_1(void){
    return double(rand()) / double(RAND_MAX); // this uses the C library random number generator.
    }
```

**C++ Code A.4: Pseudorandom numbers from an uniform \([0, 1]\) distribution**

**Exercise A.1.**

Replace the `random_uniform` function here by an alternative of higher quality, by looking into what numerical libraries is available on your computing platform, or by downloading a high quality random number generator from such places as mathlib or statlib.

These uniformly distributed distributed random variates are used as a basis for the polar method for normal densities discussed in Knuth (1997) and implemented as shown in C++ Code A.5.

```cpp
#include <cmath>
#include <cstdlib>
using namespace std;

double random_uniform_0_1(void);

double random_normal(void){
    double U1, U2, V1, V2;
    double S = 2;
    while (S >= 1) {
        U1 = random_uniform_0_1();
        U2 = random_uniform_0_1();
        V1 = 2.0*U1 - 1.0;
        V2 = 2.0*U2 - 1.0;
        S = pow(V1, 2) + pow(V2, 2);
    }
    double X1 = V1*sqrt((-2.0*sqrt(3)))/S;
    return X1;
    }
```

**C++ Code A.5: Pseudorandom numbers from a normal \((0, 1)\) distribution**

**Example**

1. Generate 5 random uniform numbers \((0, 1)\)
2. Generate 5 random N\((0, 1)\) numbers.
A.6 Cumulative probabilities for general multivariate distributions

When moving beyond the bivariate case calculation of probability integrals become more of an exercise in general numerical integration. A typical tool is Monte Carlo integration, but that is not the only possibility.

A.7 Implementation in C++11

The latest standard of the C++ language, C++11, includes a library for generating random numbers. See (Stroustrup, 1997a, Section 5.6.3).

A.8 References

Tong (1990) discusses the multivariate normal distribution, and is a good reference. For the ultimate source, see Knuth (1997).
Appendix B

C++ concepts

This chapter contains a listing of various C/C++ concepts and some notes on each of them.

**accumulate()** Function accumulating the elements of a sequence.

**bool** Boolean variable, taking on the two values true and false. For historical reasons one can also use the values zero and one for false and true.

**class** (C++ keyword).

**const** (qualifier to variable in C++ function call).

**double** (basic type). A floating point number with high accuracy.

**exp(x)** (C function). Defined in `<cmath>`. Returns the natural exponent $e$ to the given power $x$, $e^x$.

**fabs**

**float** (basic type). A floating point number with limited accuracy.

**for Loop**

**header file**

**if**

Indexation (in vectors and matrices). To access element number $i$ in an array $A$, use $A[i-1]$. Well known trap for people coming to C from other languages. Present in C for historical efficiency reasons. Arrays in C were implemented using pointers. Indexing was done by finding the first element of the array, and then adding pointers to find the indexed element. The first element is of course found by adding nothing to the first element, hence the first element was indexed by zero.

**include**

**inline** (qualifier to C++ function name). Hint to the optimizer that this function is most efficiently implemented by *Inlining* it, or putting the full code of the function into each instance of its calling. Has the side effect of making the function local to the file in which it is defined.

**int** (basic type). An integer with a limited number of significant digits.

**log(x)** (C function). Defined in `<cmath>`. Calculates the natural logarithm $\ln(x)$ of its argument.

**long** (basic type). An integer that can contain a large number.

**min_element()** (C++ function)
max_element (C++ function)
namespace (C++ concept)
return
standard namespace (C++ concept)
string (C++ basic type)
using (C++ concept)
vector (C++ container class). Defined in <vector>
while
Appendix C

Interfacing to external libraries

Contents

| C.1  | Boost                                      | 239 |
| C.2  | Newmat                                     | 239 |
| C.3  | IT++                                       | 239 |
| C.4  | GSL                                        | 239 |
| C.4.1| The evaluation of \( N_3 \)                | 239 |
| C.5  | Internet links                             | 240 |

In various places we have used routines from other public domain packages. In this appendix there are some notes about the libraries which are referenced, and some comments about linking to them.

C.1 Boost

Large collection of peer-reviewed code. Particularly useful: the date_time library.

C.2 Newmat

Newmat is a matrix library that attempts to do matlab-like operations in C++, in that one can define matrices and vectors, and then multiply them and so on without needing to explicitly call subroutines, instead writing the operations in a more mathematically oriented way.

C.3 IT++

IT++ is a large library with various different

C.4 GSL

C.4.1 The evaluation of \( N_3 \)

In the calculation of the American Put approximation of Geske and Johnson (1984), a trivariate normal needs to be evaluated. Following the discussion in the paper, this is evaluated as

\[
N_3(h, k, j; \rho_{12}, \rho_{13}, \rho_{23}) = \int_{-\infty}^{j} n(z) N_2 \left( \frac{k - \rho_{23} z}{\sqrt{1 - \rho_{23}^2}}, \frac{h - \rho_{13} z}{\sqrt{1 - \rho_{13}^2}}, \frac{\rho_{12} - \rho_{13} \rho_{23}}{\sqrt{1 - \rho_{13}^2 \sqrt{1 - \rho_{23}^2}}} \right)
\]
This is an univariate integral, and can be evaluated using quadrature.

#include <cmath>
#include <iostream>
using namespace std;

#include "gsl/gsl_integration.h"
#include "normdist.h"

struct n3_Parms { double h; double k; double rho12; double rho13; double rho23; };

extern "C"

double f3(double z, void *p){
    struct n3_Parms *parms = (struct n3_Parms*)p;
    double h = (parms->h);
    double k = (parms->k);
    double rho12 = (parms->rho12);
    double rho13 = (parms->rho13);
    double rho23 = (parms->rho23);
    double f = n(z);
    f*=N( (k-rho23*z)/sqrt(1.0-rho23*rho23),
             (h-rho13*z)/sqrt(1.0-rho13*rho13)),
             (rho12-rho13*rho23)/sqrt(1.0-rho13*rho13)*sqrt(1.0-rho23*rho23));
    return f;
};

double N3(const double & h, const double & k, const double & j,
const double & rho12, const double & rho13, const double & rho23){
    struct n3_Parms parms = { h, k, rho12, rho13, rho23};
    size_t n=1000;
    gsl_integration_workspace* w = gsl_integration_workspace_alloc(n);
    gsl_function F;
    F.function = &f;
    F.params = &parms;
    double result, error;
    gsl_integration_qags(&F,-20.0,1e-7,1e-7,n,w,&result,&error);
    return result;
};


C.5 Internet links

Some useful internet links

- [http://www.robertnz.net](http://www.robertnz.net): Homepage for Newmat, the matrix class used as an example.
Appendix D

Summarizing routine names

In many of the algorithms use is made of other routines. To simplify the matter all routines are summarised in one header file, fin_recipes.h. This appendix shows this file.

```cpp
// file: fin_recipes.h
// author: Bernt Arne Oedegaard
// defines all routines in the financial numerical recipes “book”

#ifndef FIN_RECIPES_H
#define FIN_RECIPES_H

#include <vector>
#include <cmath>

using namespace std;

///////// present value /////////////////////////
// discrete coumpounding
/////////////////////////////////
// discrete, annual compounding

double cash_flow_pv_discrete ( const vector<double>& cflow_times, const vector<double>& cflow_amounts, const double& r);

double cash_flow_irr_discrete(const vector<double>& cflow_times, const vector<double>& cflow_amounts);
bool cash_flow_unique_irr(const vector<double>& cflow_times, const vector<double>& cflow_amounts);

double bonds_price_discrete(const vector<double>& cashflow_times, const vector<double>& cashflows, const double& r);

double bonds_yield_to_maturity_discrete(const vector<double>& times, const vector<double>& amounts, const double& bondprice);

double bonds_duration_discrete(const vector<double>& times, const vector<double>& cashflows, const double& r);

double bonds_duration_macaulay_discrete(const vector<double>& cashflow_times, const vector<double>& cashflows, const double& bond_price);

double bonds_duration_modified_discrete(const vector<double>& times, const vector<double>& amounts, const double& bond_price);

double bonds_convexity_discrete(const vector<double>& times, const vector<double>& cflow_amounts, const double& r);

///////// continuous compounding

double cash_flow_pv(const vector<double>& cflow_times,const vector<double>& cflow_amounts, const double& r);

double cash_flow_irr(const vector<double>& cflow_times, const vector<double>& cflow_amounts);

double bonds_price(const vector<double>& cashflow_times, const vector<double>& cashflows, const double& r);

double bonds_price(const vector<double>& principal_times, const vector<double>& principal_amounts, const double& r);
```

241
double bonds_duration(const vector<double>& cashflow_times, const vector<double>& cashflows, const double& r);
double bonds_yield_to_maturity(const vector<double>& cashflow_times, const vector<double>& cashflows, const double& bond_price);
double bonds_duration_macaulay(const vector<double>& cashflow_times, const vector<double>& cashflows, const double& bond_price);
double bonds_convexity(const vector<double>& cashflow_times, const vector<double>& cashflows, const double& y);

// term structure basics

double term_structure_yield_from_discount_factor(const double& dfact, const double& t);
double term_structure_discount_factor_from_yield(const double& r, const double& t);
double term_structure_forward_rate_from_discount_factors(const double& d_t1, const double& d_t2, const double& time);
double term_structure_forward_rate_from_yields(const double& r_t1, const double& r_t2, const double& t1, const double& t2);
double term_structure_yield_linearly_interpolated(const double& time, const vector<double>& obs_times, const vector<double>& obs_yields);

// a term structure class

class term_structure_class{
public:
    virtual "term_structure_class();
    virtual double r(const double& t) const; // short rate, yield on zero coupon bond
    virtual double d(const double& t) const; // discount factor
    virtual double f(const double& t1, const double& t2) const; // forward rate
};

class term_structure_class_flat : public term_structure_class{
private:
    double R_;
    // interest rate
public:
    term_structure_class_flat(const double& r);
    virtual "term_structure_class_flat();
    virtual double r(const double& t) const;
    void set_int_rate(const double& r);
};

class term_structure_class_interpolated : public term_structure_class{
private:
    vector<double> times_; // use to keep a list of yields
    vector<double> yields_;  
    void clear();
public:
    term_structure_class_interpolated();
    term_structure_class_interpolated(const vector<double>& times, const vector<double>& yields);
    virtual "term_structure_class_interpolated();
    term_structure_class_interpolated operator=(const term_structure_class_interpolated&);
    term_structure_class_interpolated operator+(const term_structure_class_interpolated&);
    int no_observations() const { return int(times_.size()); }
    virtual double r(const double& T) const;
    void set_interpolated_observations(vector<double>& times, vector<double>& yields);
};

// using the term structure classes

double bonds_price(const vector<double>& cashflow_times, const vector<double>& cashflows, const term_structure_class& d);

double bonds_duration(const vector<double>& cashflow_times, const vector<double>& cashflow_amounts, const term_structure_class& d);
double bonds_convexity(const vector<double>& cashflow_times,
const vector<double>& cashflow_amounts,
const term_structure_class& d);

/// Futures pricing
double futures_price(const double& S, const double& r, const double& time_to_maturity);

/// Binomial option pricing

/// one period binomial
double option_price_call_european_binomial_single_period(const double& S, const double& K, const double& r,
const double& u, const double& d);

/// multiple period binomial
double option_price_call_european_binomial_multi_period_given_ud(const double& S, const double& K, const double& r,
const double& u, const double& d, const int& no_periods);

/// multiple period binomial
vector<vector<double>> binomial_tree(const double& S0, const double& u, const double& d,
const int& no_steps);

/// Black Scholes formula

double option_price_call_black_scholes(const double& S, const double& K, const double& r,
const double& sigma, const double& time);
double option_price_put_black_scholes(const double& S, const double& K, const double& r,
const double& sigma, const double& time);

double option_price_implied_volatility_call_black_scholes_newton(const double& S, const double& K,
const double& r, const double& time, const double& option_price);
double option_price_implied_volatility_put_black_scholes_newton(const double& S, const double& K,
const double& r, const double& time, const double& option_price);

double option_price_implied_volatility_call_black_scholes_bisections(const double& S, const double& K,
const double& r, const double& time, const double& option_price);
double option_price_implied_volatility_put_black_scholes_bisections(const double& S, const double& K,
const double& r, const double& time, const double& option_price);

double option_price_delta_call_black_scholes(const double& S, const double& K, const double& r,
const double& sigma, const double& time);
double option_price_delta_put_black_scholes(const double& S, const double& K, const double& r,
const double& sigma, const double& time);

void option_pricepartials_call_black_scholes(const double& S, const double& K, const double& r,
const double& sigma, const double& time, double& Delta, double& Gamma, double& Theta,
double& Vega, double& Rho);

void option_pricepartials_put_black_scholes(const double& S, const double& K, const double& r,
const double& sigma, const double& time, double& Delta, double& Gamma, double& Theta,
double& Vega, double& Rho);

/// warrant price
double warrant_price_adjusted_black_scholes(const double& S, const double& K,
const double& r, const double& q,
const double& sigma, const double& time,
const double& no_warrants_outstanding);

double warrant_price_adjusted_black_scholes(const double& S, const double& K,
const double& r, const double& q,
const double& sigma, const double& time,
const double& no_warrants_outstanding);
const double& no_shares_outstanding);

/// Extensions of the Black Scholes model ///////////

double option_price_european_call_payout(const double& S, const double& K, const double& r, const double& b, const double& sigma, const double& time);
double option_price_european_put_payout(const double& S, const double& K, const double& r, const double& b, const double& sigma, const double& time);
double option_price_european_call_dividends(const double& S, const double& K, const double& r, const double& sigma, const double& time, const vector<double>& dividend_times, const vector<double>& dividend_amounts);
double option_price_european_put_dividends(const double& S, const double& K, const double& r, const double& sigma, const double& time, const vector<double>& dividend_times, const vector<double>& dividend_amounts);
double option_price_american_call_one_dividend(const double& S, const double& K, const double& r, const double& sigma, const double& D1, const double& tau1);
double futures_option_price_european_black(const double& F, const double& K, const double& r, const double& sigma, const double& time);
double futures_option_price_put_european_black(const double& F, const double& K, const double& r, const double& sigma, const double& time);
double currency_option_price_european(const double& S, const double& K, const double& r, const double& sigma, const double& time);
double currency_option_price_put_european(const double& S, const double& K, const double& r, const double& sigma, const double& time);
double option_price_american_perpetual_call(const double& S, const double& K, const double& r, const double& q, const double& sigma);
double option_price_american_perpetual_put(const double& S, const double& K, const double& r, const double& q, const double& sigma);

// binomial option approximation ////////////////

double option_price_call_european_binomial(const double& S, const double& K, const double& r, const double& sigma, const double& t, const int& steps);
double option_price_put_european_binomial(const double& S, const double& K, const double& r, const double& sigma, const double& t, const int& steps);
double option_price_call_american_binomial(const double& S, const double& K, const double& r, const double& sigma, const double& t, const int& steps);
double option_price_put_american_binomial(const double& S, const double& K, const double& r, const double& sigma, const double& t, const int& steps);
double option_price_call_american_binomial(const double& S, const double& K, const double& r, const double& y, const double& sigma, const double& t, const int& steps);
double option_price_put_american_binomial(const double& S, const double& K, const double& r, const double& y, const double& sigma, const double& t, const int& steps);
double option_price_call_american_discrete_dividends_binomial(const double& S, const double& K, const double& r, const double& sigma, const double& t, const int& steps, const vector<double>& dividend_times, const vector<double>& dividend_amounts);
double option_price_put_american_discrete_dividends_binomial(const double& S, const double& K, const double& r, const double& sigma, const double& t, const int& steps, const vector<double>& dividend_times, const vector<double>& dividend_amounts);
double option_price_call_american_proportional_dividends_binomial(const double& S, const double& K, const double& r, const double& sigma, const double& y, const double& t, const int& no_steps, const vector<double>& dividend_times,
const vector<double> & dividend_yields);

double option_price_put_american_proportional_dividends_binomial(const double & S, const double & K, const double & r,
const double & sigma, const double & time, const int & no_steps,
const vector<double> & dividend_times,
const vector<double> & dividend_yields);

double option_price_delta_american_call_binomial(const double & S, const double & K, const double & r,
const double & sigma, const double & t, const int & no_steps);

double option_price_delta_american_put_binomial(const double & S, const double & K, const double & r,
const double & sigma, const double & t, const int & no_steps);

void option_price_partials_american_binomial(const double & S, const double & K, const double & r,
const double & sigma, const double & time, const int & no_steps,
double & delta, double & gamma, double & theta,
double & vega, double & rho);

void option_price_partials_american_put_binomial(const double & S, const double & K, const double & r,
const double & sigma, const double & time, const int & no_steps,
double & delta, double & gamma, double & theta,
double & vega, double & rho);

double futures_option_price_call_american_binomial(const double & F, const double & K, const double & r, const double & sigma,
const double & time, const int & no_steps);

double futures_option_price_put_american_binomial( const double & F, const double & K, const double & r, const double & sigma,
const double & time, const int & no_steps);

double currency_option_price_call_american_binomial( const double & S, const double & K, const double & r, const double & r_f,
const double & sigma, const double & t, const int & n);

double currency_option_price_put_american_binomial( const double & S, const double & K, const double & r, const double & r_f,
const double & sigma, const double & t, const int & n);

//////////////////////////////////////////////////////////////////////// finite differences //////////////////////////////////////////////////////////////////////

double option_price_call_american_finite_diff_explicit(const double & S, const double & K, const double & r,
const double & sigma, const double & time,
const int & no_S_steps, const int & no_t_steps);

double option_price_put_american_finite_diff_explicit( const double & S, const double & K, const double & r,
const double & sigma, const double & time,
const int & no_S_steps, const int & no_t_steps);

double option_price_call_european_finite_diff_explicit(const double & S, const double & K, const double & r,
const double & sigma, const double & time,
const int & no_S_steps, const int & no_t_steps);

double option_price_put_european_finite_diff_explicit( const double & S, const double & K, const double & r,
const double & sigma, const double & time,
const int & no_S_steps, const int & no_t_steps);

double option_price_call_american_finite_diff_implicit( const double & S, const double & K, const double & r,
const double & sigma, const double & time,
const int & no_S_steps, const int & no_t_steps);

double option_price_put_american_finite_diff_implicit( const double & S, const double & K, const double & r,
const double & sigma, const double & time,
const int & no_S_steps, const int & no_t_steps);

double option_price_call_european_finite_diff_implicit(const double & S, const double & K, const double & r,
const double & sigma, const double & time,
const int & no_S_steps, const int & no_t_steps);

double option_price_put_european_finite_diff_implicit( const double & S, const double & K, const double & r,
const double & sigma, const double & time,
const int & no_S_steps, const int & no_t_steps);

double option_price_put_european_finite_diff_implicit( const double & S, const double & K, const double & r,
const double & sigma, const double & time,
const int & no_S_steps, const int & no_t_steps);

double option_price_put_european_finite_diff_implicit( const double & S, const double & K, const double & r,
const double & sigma, const double & time,
const int & no_S_steps, const int & no_t_steps);

245
simulated option prices

Payoff only function of terminal price

```cpp
double option_price_call_european_simulated(const double& S, const double& K, const double& r, const double& sigma, const double& time_to_maturity, const int& no_sims);
double option_price_put_european_simulated(const double& S, const double& K, const double& r, const double& sigma, const double& time_to_maturity, const int& no_sims);
double option_price_delta_call_european_simulated(const double& S, const double& K, const double& r, const double& sigma, const double& time_to_maturity, const int& no_sims);
double option_price_delta_put_european_simulated(const double& S, const double& K, const double& r, const double& sigma, const double& time_to_maturity, const int& no_sims);
double simulate_lognormal_random_variable(const double& S, const double& r, const double& sigma, const double& time);

derivative_price_simulate_european_option_generic(const double& S, const double& K, const double& r, const double& sigma, const double& time, double payoff(const double& S, const double& K), const int& no_sims);

derivative_price_simulate_european_option_generic_with_control_variate(const double& S, const double& K, const double& r, const double& sigma, const double& time, double payoff(const double& S, const double& K), const int& no_sims);

derivative_price_simulate_european_option_generic_with_antithetic_variate(const double& S, const double& K, const double& r, const double& sigma, const double& time, double payoff(const double& S, const double& K), const int& no_sims);
```

payoffs of various options, to be used as function arguments in above simulations

```cpp
double payoff_call(const double& S, const double& K);
double payoff_put(const double& S, const double& K);
double payoff_asset_or_nothing_call(const double& S, const double& K);
```

approximated option prices

```cpp
double option_price_american_put_approximated_johnson(const double& S, const double& X, const double& r, const double& sigma, const double& time);

double option_price_american_call_approximated_baw(const double& S, const double& K, const double& r, const double& b, const double& sigma, const double& time);

double option_price_american_put_approximated_baw(const double& S, const double& K, const double& r, const double& b, const double& sigma, const double& time);

double option_price_american_put_approximated_geske_johnson(const double& S, const double& K, const double& r, const double& sigma, const double& time);

double option_price_american_call_approximated_bjerksund_stensland(const double& S, const double& X, const double& r, const double& q, const double& sigma, const double& time);
```

246
double option_price_american_put_approximated_bjerksund_stensland(const double& S, 
    const double& X, 
    const double& r, 
    const double& q, 
    const double& sigma, 
    const double& T);

//-------------------------------------------------------------------------------------------------------
// path dependent and other exotic options  
//-------------------------------------------------------------------------------------------------------

double option_price_call_bermudan_binomial(const double& S, const double& K, const double& r, 
    const double& q, const double& sigma, const double& time, 
    const vector<double>& potential_exercise_times, 
    const int& steps);

double option_price_put_bermudan_binomial(const double& S, const double& K, const double& r, 
    const double& q, const double& sigma, const double& time, 
    const vector<double>& potential_exercise_times, 
    const int& steps);

double option_price_european_lookback_call(const double& S, const double& Smin, const double& r, 
    const double& q, const double& sigma, const double& time); 

double option_price_european_lookback_put(const double& S, const double& Smin, const double& r, 
    const double& q, const double& sigma, const double& time);

double option_price_asian_geometric_average_price_call(const double& S, const double& K, const double& r, 
    const double& q, const double& sigma, const double& time); 

vector<double> simulate_lognormally_distributed_sequence(const double& S, const double& r, 
    const double& sigma, const double& time, const int& no_steps);

double derivative_price_simulate_european_option_generic(const double& S, const double& K, const double& r, 
    const double& sigma, const double& time, 
    double payoff(const vector<double>& S, 
        const double& K), 
    const int& no_steps, const int& nosims);

double derivative_price_simulate_european_option_generic_with_control_variate(const double& S, const double& K, 
    const double& r, const double& sigma, 
    const double& time, 
    double payoff(const vector<double>& S, 
        const double& K), 
    const int& no_steps, const int& nosims);

//-------------------------------------------------------------------------------------------------------
// payoffs of various options, to be used as function arguments in above simulations

double payoff_arithmetic_average_call(const vector<double>& prices, const double& K);

double payoff_geometric_average_call(const vector<double>& prices, const double& K);

double payoff_lookback_call(const vector<double>& prices, const double& unused_variable);

double payoff_lookback_put(const vector<double>& prices, const double& unused_variable);

//-------------------------------------------------------------------------------------------------------
// generic binomial trees

double option_price_generic_binomial(const double& S, const double& K, 
    double generic_payoff(const double& S, const double& K), 
    const double& r, const double& sigma, const double& t, const int& steps);

double payoff_binary_call(const double& S, const double& K);

double payoff_binary_put(const double& S, const double& K);
double option_price_call_american_trinomial( const double& S, const double& K, const double& r, const double& q, const double& sigma, const double& t, const int& steps);

double option_price_put_american_trinomial( const double& S, const double& K, const double& r, const double& q, const double& sigma, const double& t, const int& steps);

double heston_call_option_price(const double& S, const double& K, const double& r, const double& v, const double& rho, const double& kappa, const double& lambda, const double& theta, const double& sigma);

double bond_option_price_call_black_scholes(const double& B, const double& K, const double& r, const double& sigma, const double& time);

double bond_option_price_put_black_scholes(const double& B, const double& K, const double& r, const double& delta);

double bond_option_price_call_coupon_black_scholes(const double& B, const double& K, const double& r, const double& sigma, const double& time);
const double & a, const double & b, const double & sigma);

/// defining classes wrapping the above term structure approximations

class term_structure_class_nelson_siegel : public term_structure_class {
private:
  double beta0_, beta1_, beta2_, lambda_; 
public:
  term_structure_class_nelson_siegel(const double & beta0, const double & beta1, const double & beta2, const double & lambda);
  virtual double r(const double & t) const;
};

class term_structure_class_svensson : public term_structure_class {
private:
  double beta0_, beta1_, beta2_, beta3_, tau1_, tau2_; 
public:
  term_structure_class_svensson(const double & beta0, const double & beta1, const double & beta2, const double & beta3, const double & tau1, const double & tau2);
  virtual double r(const double & T) const;
};

class term_structure_class_cubic_spline : public term_structure_class {
private:
  double b_, c_, d_; vector<double> f_; vector<double> knots_; 
public:
  term_structure_class_cubic_spline(const double & b, const double & c, const double & d, const vector<double> & f, const vector<double> & knots);
  virtual double d(const double & t) const; // discount factor
};

class term_structure_class_cir : public term_structure_class {
private:
  double r_, kappa_, lambda_, theta_, sigma_; 
public:
  term_structure_class_cir(const double & r, const double & k, const double & l, const double & th, const double & sigma);
  virtual double d(const double & t) const; // discount factor
};

class term_structure_class_vasicek : public term_structure_class {
private:
  double r_, a_, b_, sigma_; 
public:
  term_structure_class_vasicek(const double & r, const double & a, const double & b, const double & sigma);
  virtual double d(const double & T) const;
};

// binomial term structure models
// bond option, rendleman bartter (binomial)

double bond_option_price_call_zero_american_rendleman_bartter(const double & K, const double & option_maturity, const double & S, const double & M, const double & interest, const double & bond_maturity, const double & maturity_payment, const int & no_steps);

vector<vector<double>> interest_rate_trees_gbm_build(const double & r0, const double & u, const double & d, const int & n);

double interest_rate_trees_gbm_value_of_cashflows(const vector<vector<double>> & cflow, const vector<vector<double>> & r_tree,
const double q);

double interest_rate_trees_gbm_value_ofCallableBond(const vector<double>& cflows,
            const vector< vector<double> >& r_tree,
            const double& q,
            const int& first_call_time,
            const double& call_price);

double priceEuropeanCallOptionOnBondUsingHoLee(term_structure_class* initial,
            const double& delta,
            const double& pi,
            const vector<double>& underlyingBondCflowTimes,
            const vector<double>& underlyingBondCflows,
            const double& K,
            const double& optionTimeToMaturity);

///////////////////////////
// ho and lee modelling

class term_structure_class_ho_lee : public term_structure_class {
private:
    term_structure_class* initial_term_
    int n_,
    int i_,
    double delta_,
    double pi_;
public:
    term_structure_class_ho_lee(term_structure_class* fitted_term,
        const int & n,
        const int & i,
        const double& lambda,
        const double& pi);

    double d(const double& T) const;
};

vector< vector<term_structure_class_ho_lee> >
term_structure_ho_lee_build_term_structure_tree(term_structure_class* initial,
        const int& no_steps,
        const double& delta,
        const double& pi);

double priceEuropeanCallOptionOnBondUsingHoLee(term_structure_class* initial,
            const double& delta,
            const double& pi,
            const vector<double>& underlyingBondCflowTimes,
            const vector<double>& underlyingBondCflows,
            const double& K,
            const double& optionTimeToMaturity);

///////////////////////////
// term structure derivatives, analytical solutions

double bond_option_price_call_zero_vasicek(const double& X, const double& r,
            const double& optionTimeToMaturity,
            const double& bondTimeToMaturity,
            const double& a, const double& b, const double& sigma);

double bond_option_price_put_zero_vasicek(const double& X, const double& r,
            const double& optionTimeToMaturity,
            const double& bondTimeToMaturity,
            const double& a, const double& b, const double& sigma);

#endif
Appendix E

Installation

The routines discussed in the book are available for download.

E.1 Source availability

The algorithms are available from my home page as a ZIP file containing the source code. These have been tested with the latest version of the GNU C++ compiler. As the algorithms in places uses code from the Standard Template Library, other compilers may not be able to compile all the files directly. If your compiler complains about missing header files you may want to check if the STL header files have different names on your system. The algorithm files comply with the current ANSI standard for C++ libraries. If the compiler is more than a couple of years old, it will not have STL. Alternatively, the GNU compiler gcc is available for free on the internet, for most current operating systems.
# List of C++ Codes

1.1 A complete program ................................................................. 9
1.2 Basic operations for the date class ........................................... 12
1.3 Comparison operators for the date class ....................................... 13
1.4 Iterative operators for the date class ......................................... 14
3.1 Present value with discrete compounding ..................................... 27
3.2 Estimation of the internal rate of return ..................................... 31
3.3 Test for uniqueness of IRR ....................................................... 33
3.4 Present value calculation with continuously compounded interest ........ 35
4.1 Bond price calculation with discrete, annual compounding ............... 38
4.2 Bond yield calculation with discrete, annual compounding ............... 39
4.3 Bond duration using discrete, annual compounding and a flat term structure ... 41
4.4 Calculating the Macaulay duration of a bond ................................ 42
4.5 Modified duration ................................................................. 43
4.6 Bond convexity with a flat term structure and annual compounding ...... 44
4.7 Bond price calculation with continuously compounded interest and a flat term structure ... 47
4.8 Bond duration calculation with continuously compounded interest and a flat term structure ... 48
4.9 Calculating the Macaulay duration of a bond with continuously compounded interest and a flat term structure ... 48
4.10 Bond convexity calculation with continuously compounded interest and a flat term structure ... 48
5.1 Term structure transformations .................................................. 53
5.2 Default code for transformations between discount factors, spot rates and forward rates in a term structure class .......................................................... 56
5.3 Implementing term structure class using a flat term structure .......... 57
5.4 Interpolated term structure from spot rates ................................... 60
5.5 Term structure class using linear interpolation between spot rates ....... 62
5.6 Pricing a bond with a term structure class .................................. 64
5.7 Calculating a bonds duration with a term structure class .................. 65
5.8 Calculating a bonds convexity with a term structure class ................. 65
6.1 Mean variance calculations using IT++ ....................................... 77
6.2 Mean variance calculations using Newmat ................................... 78
6.3 Calculating the unconstrained frontier portfolio given an expected return using Newmat .... 79
6.4 Calculating the unconstrained frontier portfolio given an expected return using IT++ ...... 80
7.1 Futures price ............................................................................. 81
8.1 Binomial European, one period ..................................................... 84
8.2 Building a binomial tree ............................................................. 87
8.3 Binomial multiperiod pricing of European call option ....................... 88
9.1 Price of European call option using the Black Scholes formula .......... 90
9.2 Calculating the delta of the Black Scholes call option price ................ 94
9.3 Calculating the partial derivatives of a Black Scholes call option ........ 95
9.4 Calculation of implied volatility of Black Scholes using bisections ....... 96
9.5 Calculation of implied volatility of Black Scholes using Newton-Raphson .... 97
10.1 Adjusted Black Scholes value for a Warrant .................................. 101
11.1 Option price, continuous payout from underlying ............................. 103
11.2 European option price, dividend paying stock ................................ 104
11.3 Option price, Roll–Geske–Whaley call formula for dividend paying stock ............................................................................................. 107
11.4 Price of European Call option on Futures contract ............................ 108
23.1 RB binomial model for European call on zero coupon bond ........................................... 212
24.1 Building interest rate tree .................................................................................................. 215
24.2 Valuing cash flows .......................................................................................................... 217
24.3 Valuing callable bond ....................................................................................................... 218
25.1 Term structure class for Ho-Lee ....................................................................................... 222
25.2 Term structure class for Ho-Lee, calculation of discount function ................................. 223
25.3 Building a term structure tree ......................................................................................... 223
25.4 Pricing of European call option on straight bond using Ho-Lee ................................. 225
26.1 Bond option pricing using the Vasicek model .............................................................. 228
A.1 The normal distribution function ..................................................................................... 231
A.2 The cumulative normal ...................................................................................................... 232
A.3 Approximation to the cumulative bivariate normal ........................................................ 234
A.4 Pseudorandom numbers from an uniform [0, 1] distribution ........................................ 235
A.5 Pseudorandom numbers from a normal (0, 1) distribution ............................................ 235
C.1 Approximating \( N_a() \) using the method of Geske and Johnson (1984) ..................... 240
# List of Matlab Codes

<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1</td>
<td>Calculation of minimum variance portfolio for given return</td>
<td>71</td>
</tr>
<tr>
<td>6.2</td>
<td>Sharpe Ratio</td>
<td>76</td>
</tr>
<tr>
<td>6.3</td>
<td>Treynor Ratio</td>
<td>76</td>
</tr>
<tr>
<td>6.4</td>
<td>Jensens alpha</td>
<td>76</td>
</tr>
<tr>
<td>9.1</td>
<td>Price of European call option using the Black Scholes formula</td>
<td>91</td>
</tr>
<tr>
<td>12.1</td>
<td>Price of American call using a binomial approximation</td>
<td>116</td>
</tr>
<tr>
<td>12.2</td>
<td>Price of American put using a binomial approximation</td>
<td>117</td>
</tr>
<tr>
<td>13.1</td>
<td>Explicit finite differences calculation of American put option</td>
<td>135</td>
</tr>
<tr>
<td>13.2</td>
<td>Calculation of price of American put using implicit finite differences</td>
<td>137</td>
</tr>
<tr>
<td>18.1</td>
<td>Price of american put using a trinomial tree</td>
<td>185</td>
</tr>
<tr>
<td>24.1</td>
<td>Building interest rate tree</td>
<td>215</td>
</tr>
</tbody>
</table>
Appendix F

Acknowledgements.

After this paper was put up on the net, I’ve had quite a few emails about them. Some of them has pointed out bugs and other inaccuracies. Among the ones I want to say thanks to for making improving suggestions and pointing out bugs are:

Ariel Almedal
Andrei Bejenari
Steve Bellantoni
Jean-Paul Beveraggi
Lars Gregori
Daniel Herlemont
Lorenzo Isella
Jens Larsson
Garrick Lau
Steven Leadbeater
Michael L Locher
Lotti Luca, Milano, Italy
Tuan Nguyen
Michael R Wayne
Index

\( \gamma \), 94
\( \rho \), 94
\( \theta \), 94
A, 137
a, 135
antithetic variates, 148
asset or nothing call, 151
Barone–Adesi and Whaley, 159
binary option, 151
binomial option price, 82, 112
binomial term structure models, 211
binomial_tree, 87
Black
futures option, 108
Black Scholes option pricing formula, 89
Bond
Price
Flat term structure, 36
Promised payments, 36
bond
duration, 41
price, 37
yield to maturity, 38
bond convexity, 44
bond option
basic binomial, 193
Black Scholes, 191
Vasicek, 227
bond_option_price_call_zero_american_rendleman_bartter, 212, 213, 249
date::date, 12
date::day, 12
date::month, 12
date::valid, 12
delta, 93
d, 57, 63, 249
d1, 158
d2, 158
d2, 158
build_time_series_of_bond_time_contingent_cash_flows, 225
calcP2, 158
call option, 82
cash flow, 26
cash or nothing call, 151
cash_flow_irr_discrete, 31
cash_flow_pv, 35
cash_flow_pv_discrete, 27, 28, 32, 241
cash_flow_unique_irr, 33
class, 9
cmath, 7
Complex numbers
in C++, 189
currency
option, 109
currency option
American, 130
European, 109
currency_option_price_call_american_binomial, 130
currency_option_price_call_european, 109, 110
currency_option_price_call_zero_black_scholes, 191, 194
cphony (C++ type), 6
discount factor, 26
double (C++ type), 6
duration, 41
duration, modified discrete, 45
duration_macaulay, 48
duration_macaulay_discrete, 42
duration_modified_discrete, 43
duration_to_maturity_discrete, 39, 40
true (C++ type), 6
build_time_series_of_bond_time_contingent_cash_flows,
Macaulay, 41
modified, 43

e, 78, 79
early exercise premium, 159
ex_date_time, 230
exp, 196
exp() (C++ statement), 7
explicit finite differences, 132

f, 234
f3, 240
finite differences, 132, 137
explicit, 132
for (C++ statement), 8
function prototypes (C++ concept), 146
futures
option, 108
futures_option_price_call_american_binomial, 128, 129
futures_option_price_call_european_black, 108
futures_option_price_put_european_black, 108
futures_price, 81
gamma, 94
general Brownian motion, 93
Geske and Johnson, 156

hedging parameters
Black Scholes, 93
heston_call_option_price, 189, 190
heston_integrand_j, 190
heston_Pj, 190
hT, 223

if, 13
implied volatility
calculation, 96
include (C++ statement), 7
int (C++ type), 6
interest_rate_trees_gbm_build, 215
interest_rate_trees_gbm_value_of_callable_bond, 218
interest_rate_trees_gbm_value_of_cashflows, 217, 220
internal rate of return, 30
Internet links, 240
irr, 30
iteration operator
definition of (C++ concept), 14

Jump Diffusion, 186

Links
Internet, 240
long (C++ type), 6
lookback option, 170

main, 9
Merton
Jump Diffusion, 186
modified duration, 43

Monte Carlo
antithetic variates, 148
monte carlo
control variates, 147
mv_calculate_mean, 77, 78
mv_calculate_portfolio_given_mean_unconstrained, 79, 80
mv_calculate_std_dev, 77, 78
mv_calculate_variance, 77, 78

N, 232–234
n, 231
N3, 240
next_date, 14
no_observations, 61, 242
Normal distribution
Simulation, 235
normal distribution approximations, 231

option
call, 82
currency, 109
futures, 108
lookback, 170
put, 82
Option price
Black Scholes, 89
option price
binomial, 112
simulated, 142
option_price_american_call_approximated_baw, 160, 161
option_price_american_call_approximated_bjerksund_stensland, 164
option_price_american_call_one_dividend, 106, 107
option_price_american_perpetual_call, 110
option_price_american_put_approximated_bjerksund_stensland, 164
option_price_american_put_approximated_geske_johnson, 158
option_price_american_put_approximated_johnson, 154, 155
option_price_asian_geometric_average_price_call, 169
option_price_call_american_binomial, 115, 123, 128, 178
option_price_call_american_discrete_dividends_binomial, 127
option_price_call_american_proportional_dividends_binomial, 125
option_price_call_black_scholes, 90, 92, 144, 147, 150
option_price_call_european_binomial, 114, 118
option_price_call_european_binomial_multi_period_given_ud, 88
option_price_call_european_binomial_single_period, 84, 87
option_price_call_european_simulated, 144
option_price_call_merton_jump_diffusion, 187

258
option_price_delta_american_call_binomial, 120
option_price_delta_call_black_scholes, 94, 145
option_price_delta_call_european_simulated, 145
option_price_generic_binomial, 182
option_price_european_call_dividends, 104
option_price_european_call_payout, 103, 104
option_price_european_lookback_call, 170, 171
option_price_implied_volatility_call_black_scholes, 96
option_price_implied_volatility_call_black_scholes_newton, 96
option_price_partials_american_call_binomial, 121, 122
option_price_partials_call_black_scholes, 95
option_price_put_american_finite_diff_explicit, 134
option_price_put_american_finite_diff_implicit, 138
option_price_put_american_finite_diff_implicit_itpp, 139
option_price_put_american_trinomial, 184, 185
option_price_put_bermudan_binomial, 167, 168
option_price_put_black_scholes, 141
option_price_put_european_finite_diff_explicit, 133, 136
option_price_put_european_finite_diff_implicit, 140
option_price_put_black_scholes, 95
partial derivatives
  binomial, 120
  Black Scholes, 93
partials
  Black Scholes, 93
payoff_arithmetic_average_call, 174
payoff_asset_or_nothing_call, 151
payoff_binary_call, 181
payoff_binary_put, 146, 180
payoff_cash_or_nothing_call, 151
payoff_geometric_average_call, 174
payoff_lookback_call, 174
payoff_lookback_put, 174
payoff_put, 146, 180
phi, 164
pow() (C++ statement), 7
power, 9
present_value, 26
previous_date, 14
price_european_call_option_on_bond_using_ho_lee, 225
prices, 116, 117
pricing
  relative, 82
  put option, 82
quadratic approximation, 159
r, 55, 57, 61, 242
random_normal, 235
random_uniform_0_1, 235
relativepricing, 82
Rendleman and Bartter model, 211
return
  internal rate of, 30
  rho, 94
sgn, 234
Sharpe Ratio, 75
simulate_lognormal_random_variable, 143
simulate_lognormally_distributed_sequence, 173
Simulation
  Random normal numbers, 235
  simulation, 142
string (C++ type), 6
term structure derivatives, 227
Term structure model
  Cox Ingersoll Ross, 205
term structure models
  binomial, 211
term_structure_class::d, 56
term_structure_class::f, 56
term_structure_class::r, 56
term_structure_class_cir, 206, 249
term_structure_class_cir::d, 206
term_structure_class_cir::term_structure_class_cir, 206
term_structure_class_cubic_spline, 202, 203, 249
term_structure_class_cubic_spline::d, 203
term_structure_class_cubic_spline::flat::x, 57
term_structure_class_ho_lee, 222
term_structure_class_ho_lee::d, 223
term_structure_class_ho_lee::term_structure_class_ho_lee, 222
term_structure_class_interpolated::clear, 62
term_structure_class_interpolated::r, 62
term_structure_class_interpolated::set_interpolated_observations, 62
term_structure_class_interpolated::term_structure_class_interpolated, 62
term_structure_class_nelson_siegel, 198
term_structure_class_nelson_siegel::r, 199
term_structure_class_nelson_siegel::term_structure_class_nelson_siegel, 199
term_structure_class_svensson, 201, 249
term_structure_class_svensson::r, 201
term_structure_class_svensson::term_structure_class_svensson, 201
term_structure_class_vasicek::d, 209
term_structure_class_vasicek::term_structure_class_vasicek, 209
term_structure_discount_factor_cir, 205, 207
term_structure_discount_factor_cubic_spline, 202, 204
term_structure_discount_factor_from_yield, 53
term_structure_discount_factor_vasicek, 208, 210
term_structure_forward_rate_from_discount_factors, 53, 54
term_structure_forward_rate_from_yields, 53
term_structure_ho_lee_build_term_structure_tree, 222, 223, 250
259
term_structure_yield_from_discount_factor, 53
term_structure_yield_linearly_interpolated, 60
term_structure_yield_nelson_siegel, 198, 200
term_structure_yield_svensson, 201
theta, 94
time
  value of, 26
time_contingent_cash_flows, 225
underlying security, 89
valid, 10
Vasicek, 208
vega, 94
volatility
  implied, 96
warrant_price_adjusted_black_scholes, 101
Bibliography


261


